Actuarial Science with R

1. life insurance & actuarial notations

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joint work with **Christophe Dutang** & **Vincent Goulet**
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Quantact

Meielisalp 2012 Conference, June

6th R/Rmetrics Meielisalp Workshop & Summer School
on Computational Finance and Financial Engineering

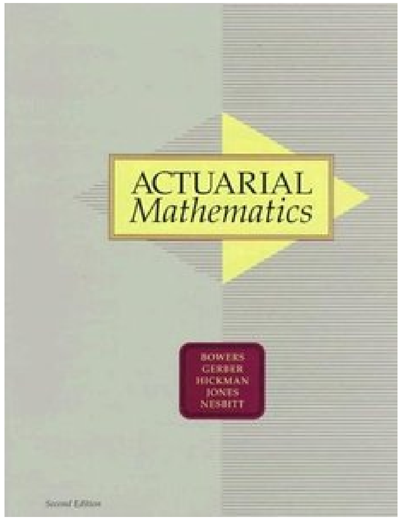
Some (standard) references

Bowers, N.L., Gerber, H.U., Hickman, J.C.,

Jones, D.A. & Nesbitt , C.J. (1997)

Actuarial Mathematics

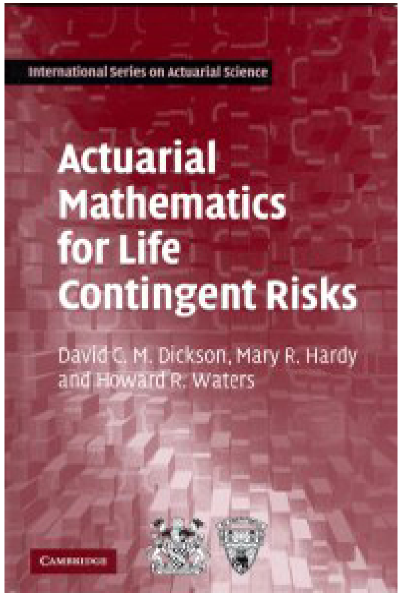
Society of Actuaries



Dickson, D.C., Hardy, M.R. & Waters, H.R. (2010)

Actuarial Mathematics for Life Contingent Risks

Cambridge University Press



Modeling future lifetime

Let (x) denote a life aged x, with $x \geq 0$.

The future lifetime of (x) is a continuous random variable T_x

Let F_x and $\overline{F}_x$ (or S_x) denote the cumulative distribution function of T_x and the survival function, respectively,

$$F_x(t) = \mathbb{P}(T_x \leq t) \text{ and } \overline{F}_x(t) = \mathbb{P}(T_x > t) = 1 - F_x(t).$$

Let μ_x denote the force of mortality at age x (or *hazard rate*),

$$\mu_x = \lim_{h\downarrow 0} \frac{\mathbb{P}(T_0 \leq x + h | T_0 > x)}{h} = \lim_{h\downarrow 0} \frac{\mathbb{P}(T_x \leq h)}{h} = \frac{-1}{\overline{F}_0(x)} \frac{d\overline{F}_0(x)}{dx} = -\frac{d \log \overline{F}_0(x)}{dx}$$

or conversely,

$$\overline{F}_x(t) = \frac{\overline{F}_0(x+t)}{\overline{F}(x)} = \exp\left(-\int_x^{x+t} \mu_s ds\right)$$

Modeling future lifetime

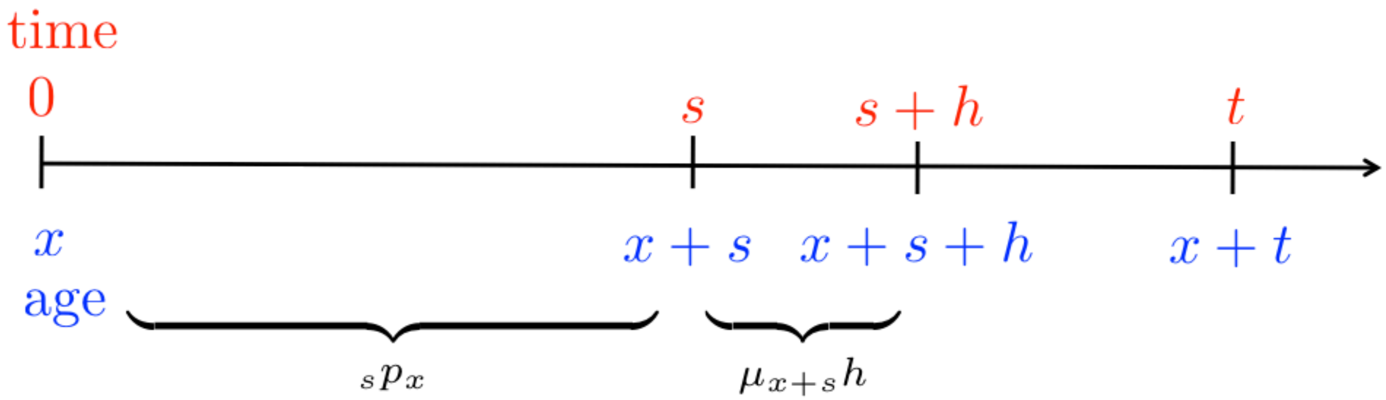
Define ${}_tp_x = \mathbb{P}(T_x > t) = \overline{F}_x(t)$ and ${}_tq_x = \mathbb{P}(T_x \leq t) = F_x(t)$, and

$$ {}_{t|h}q_x = \mathbb{P}(t < T_x \leq t + h) = {}_tp_x - {}_{t+h}p_x $$

the defered mortality probability. Further, $p_x = {}_1p_x$ and $q_x = {}_1q_x$.

Several equalities can be derived, e.g.

$$ {}_tq_x = \int_0^t {}_sp_x \mu_{x+s} ds. $$



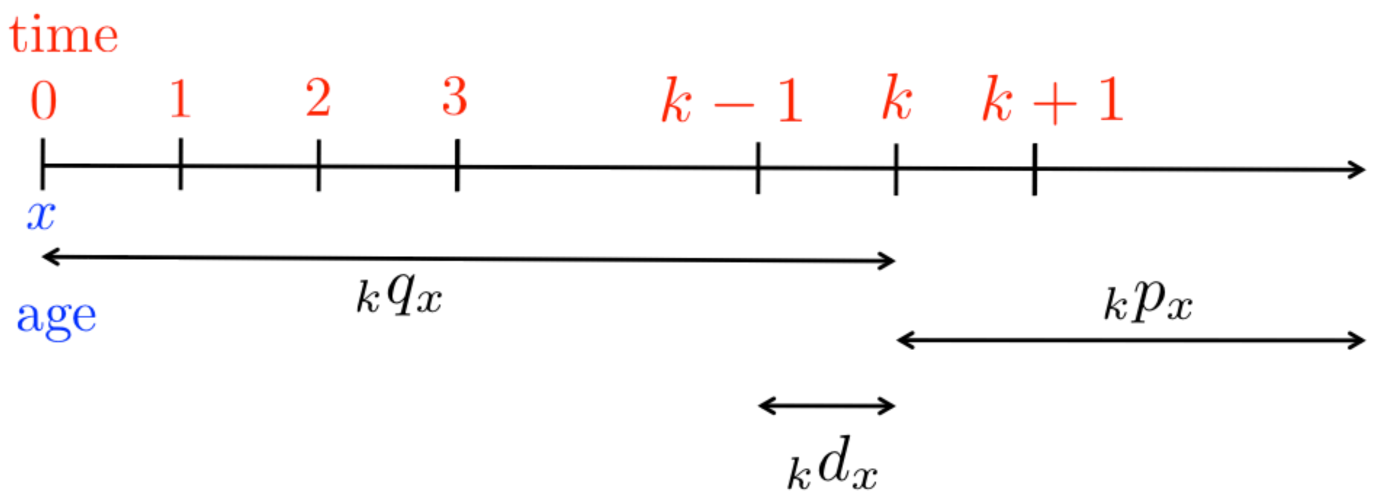
Modeling curtate future lifetime

The curtate future lifetime of (x) is the number of future years completed by (x) priors to death, $K_x = \lfloor T_x \rfloor$.

Its probability function is

$$_kd_x = \mathbb{P}(K_x = k) = {}_{k+1}q_x - {}_kq_x = {}_{k|}q_x$$

for $k \in \mathbb{N}$, and it cumulative distribution function is $\mathbb{P}(K_x \leq k) = {}_{k+1}q_x$.



Modeling future lifetime

Define the (complete) expectation of life,

$$\mathring{e}_x = \mathbb{E}(T_x) = \int_0^\infty \overline{F}_x(t)dt = \int_0^\infty {}_tp_x dt$$

and its discrete version, curtate expectation of life

$$e_x = \mathbb{E}(\lfloor T_x \rfloor) = \sum_{k=1}^\infty {}_tp_x$$

Life tables

Given x_0 (initial age, usually $x_0 = 0$), define a function l_x where $x \in [x_0, \omega]$ as

$$l_{x_0+t} = l_{x_0} \cdot {}_tp_{x_0}$$

Usually $l_0 = 100,000$. Then

$${}_tp_x = \frac{l_{x+t}}{l_x}$$

Remark : some kind of Markov property,

$${}_{k+h}p_x = \frac{L_{x+k+h}}{L_x} = \frac{L_{x+k+h}}{L_{x+k}} \cdot \frac{L_{x+k}}{L_x} = {}_hp_{x+k} \cdot {}_kp_x$$

Let $d_x = l_x - l_{x+1} = l_x \cdot q_x$

(old) French life tables

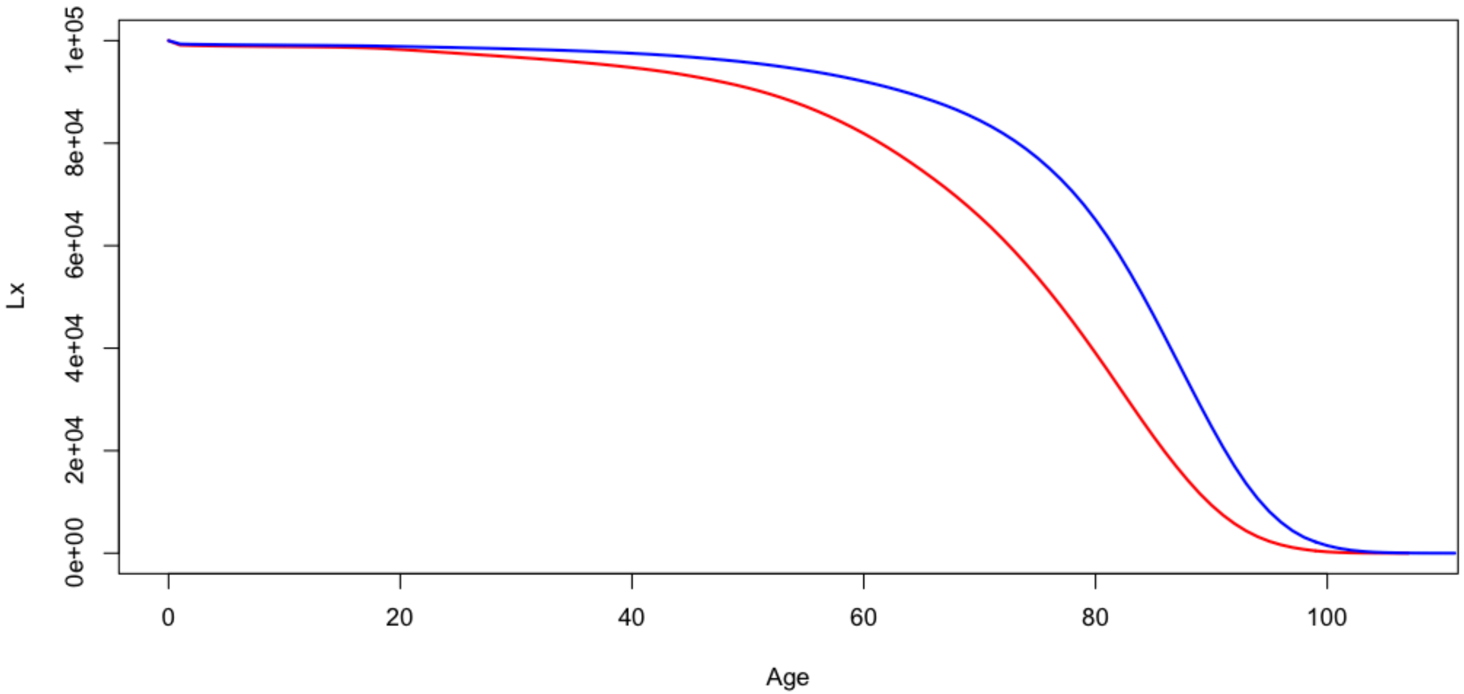
```
> TD[39:52,]
   Age    Lx
39  38 95237
40  39 94997
41  40 94746
42  41 94476
43  42 94182
44  43 93868
45  44 93515
46  45 93133
47  46 92727
48  47 92295
49  48 91833
50  49 91332
51  50 90778
52  51 90171
```

TABLE I.

Ages par années.	Survivans selon M. Halley.	N'ayant pas eu la pet. vérole.	Ayant eu la pet. vérol.	Prenant la pet. vérole pendant ch. année.	Morts de la pet. vérole pendant chaq. ann.	Somme des morts de la pet. vérole.	Morts par d'autres maladies pend. chaq. année.
0	1300	1300	0				
1	1000	896	104	137	17,1	17,1	283
2	855	685	170	99	12,4	29,5	133
3	798	571	227	78	9,7	39,2	47
4	760	485	275	66	8,3	47.5	30
5	732	416	316	56	7,0	54.5	21
6	710	359	351	48	6,0	60,5	16
7	692	311	381	42	5,2	65,7	12,8
8	680	272	408	36	4.5	70,2	7.5
9	670	237	433	32	4,0	74,2	6
10	661	208	453	28	3.5	77,7	5.5
11	653	182	471	24.4	3,0	80,7	5
12	646	160	486	21,4	2,7	83.4	4.3
13	640	140	500	18,7	2,3	85,7	3.7
14	634	123	511	16,6	2,1	87,8	3.9
15	628	108	520	14.4	1,8	89,6	4.2
16	622	94	528	12,6	1,6	91,2	4.4
17	616	83	533	11,0	1,4	92,6	4,6
18	610	72	538	9,7	1,2	93,8	4.8
19	604	63	541	8,4	1,0	94,8	5
20	598	56	542	7.4	0,9	95.7	5.1
21	592	48,5	543	6,5	0,8	96,5	5.2
22	586	42,5	543	5,6	0,7	97,2	5.3
23	579	37	542	5,0	0,6	97,8	6,4
24	572	32.4	540	4.4	0,5	98,3	6,5

(old) French life tables

```
> plot(TD$Age,TD$Lx,lwd=2,col="red",type="l",xlab="Age",ylab="Lx")
> lines(TV$Age,TV$Lx,lwd=2,col="blue")
```



Playing with life tables

From life tables, it is possible to derive probabilities, e.g. ${}_{10}p_{40} = \mathbb{P}(T_{40} > 10)$

```
> TD$Lx[TD$Age==50]
[1] 90778
> TD$Lx[TD$Age==40]
[1] 94746
> x <- 40
> h <- 10
> TD$Lx[TD$Age==x+h]/TD$Lx[TD$Age==x]
[1] 0.9581196
> TD$Lx[x+h+1]/TD$Lx[x+1]
[1] 0.9581196
```

Defining matrices $\boldsymbol{P}=[{}_kp_x]$, $\boldsymbol{Q}=[{}_kq_x]$ and $\boldsymbol{D}=[{}_kd_x]$

For $k=1,2,\cdots$ and $x=0,1,2,\cdots$ it is possible to calculate ${}_kp_x$. If $x\in\mathbb{N}_*$, define $\boldsymbol{P}=[{}_kp_x]$.

```
> Lx <- TD$Lx
> m <- length(Lx)
> p <- matrix(0,m,m); d <- p
> for(i in 1:(m-1)){
+ p[1:(m-i),i] <- Lx[1+(i+1):m]/Lx[i+1]
+ d[1:(m-i),i] <- (Lx[(1+i):(m)]-Lx[(1+i):(m)+1])/Lx[i+1]}
> diag(d[(m-1):1,]) <- 0
> diag(p[(m-1):1,]) <- 0
> q <- 1-p
```

Here, `p[10,40]` corresponds to ${}_{10}p_{40}$:

```
> p[10,40]
[1] 0.9581196
```

Remark : matrices will be more convenient than functions for computations...

Working with matrices $P = [{}_kp_x]$ and $Q = [{}_kq_x]$

(Curtate) expactation of life is

$$e_x = \mathbb{E}(K_x) = \sum_{k=1}^{\infty} k \cdot {}_{k|1}q_x = \sum_{k=1}^{\infty} {}_kp_x$$

```
> x <- 45
> S <- p[,45]/p[1,45]
> sum(S)
[1] 30.46237
```

It is possible to define a function

```
> life.exp=function(x){sum(p[1:nrow(p),x])}
> life.exp(45)
[1] 30.32957
```

Insurance benefits and expected present value

Let i denote a (constant) interest rate, and $\nu = (1+i)^{-1}$ the discount factor.

Consider a series of payments $\boldsymbol{C} = (C_1, \cdots, C_k)$ due with probability $\boldsymbol{p} = (p_1, \cdots, p_k)$, at times $\boldsymbol{t} = (t_1, \cdots, t_k)$. The expected present value of those benefits is

$$\sum_{j=1}^{k} \frac{C_j \cdot p_j}{(1+i)^{t_j}} = \sum_{j=1}^{k} \nu^{t_j} \cdot C_j \cdot p_j$$

Consider here payments at dates $\{1, 2, \cdots, k\}$.

Insurance benefits and expected present value

Example : Consider a whole life insurance, for some insured aged x, where benefits are payables following the death, if it occurs with k years from issue, i.e. $p_j = {}_j d_x$,

$$\sum_{j=1}^{n} \frac{C \cdot \mathbb{P}(K_x = j)}{(1+i)^j} = C \cdot \sum_{j=1}^{n} \nu^j \cdot {}_{j|}q_x.$$

```
>  k <- 20; x <- 40; i <- 0.03
>  C <- rep(100,k)
>  P <- d[1:k,x]
>  sum((1/(1+i)^(1:k))*P*C)
[1] 9.356656
>  sum(cumprod(rep(1/(1+i),k))*P*C)
[1] 9.356656
```

Insurance benefits and expected present value

Example : Consider a temporary life annuity-immediate, where benefits are paid at the end of the year, as long as the insured (x) survives, for up a total of k years (k payments)

$$\sum_{j=1}^{n} \frac{C \cdot \mathbb{P}(K_x = j)}{(1+i)^j} = C \sum_{j=1}^{n} \nu^j \cdot {}_j p_x.$$

```
>  k <- 20; x <- 40; i <- 0.03
>  C <- rep(100,k)
>  P <- p[1:k,x]
>  sum((1/(1+i)^(1:k))*P*C)
[1] 1417.045
>  sum(cumprod(rep(1/(1+i),k))*P*C)
[1] 1417.045
```

it is possible to define a general function

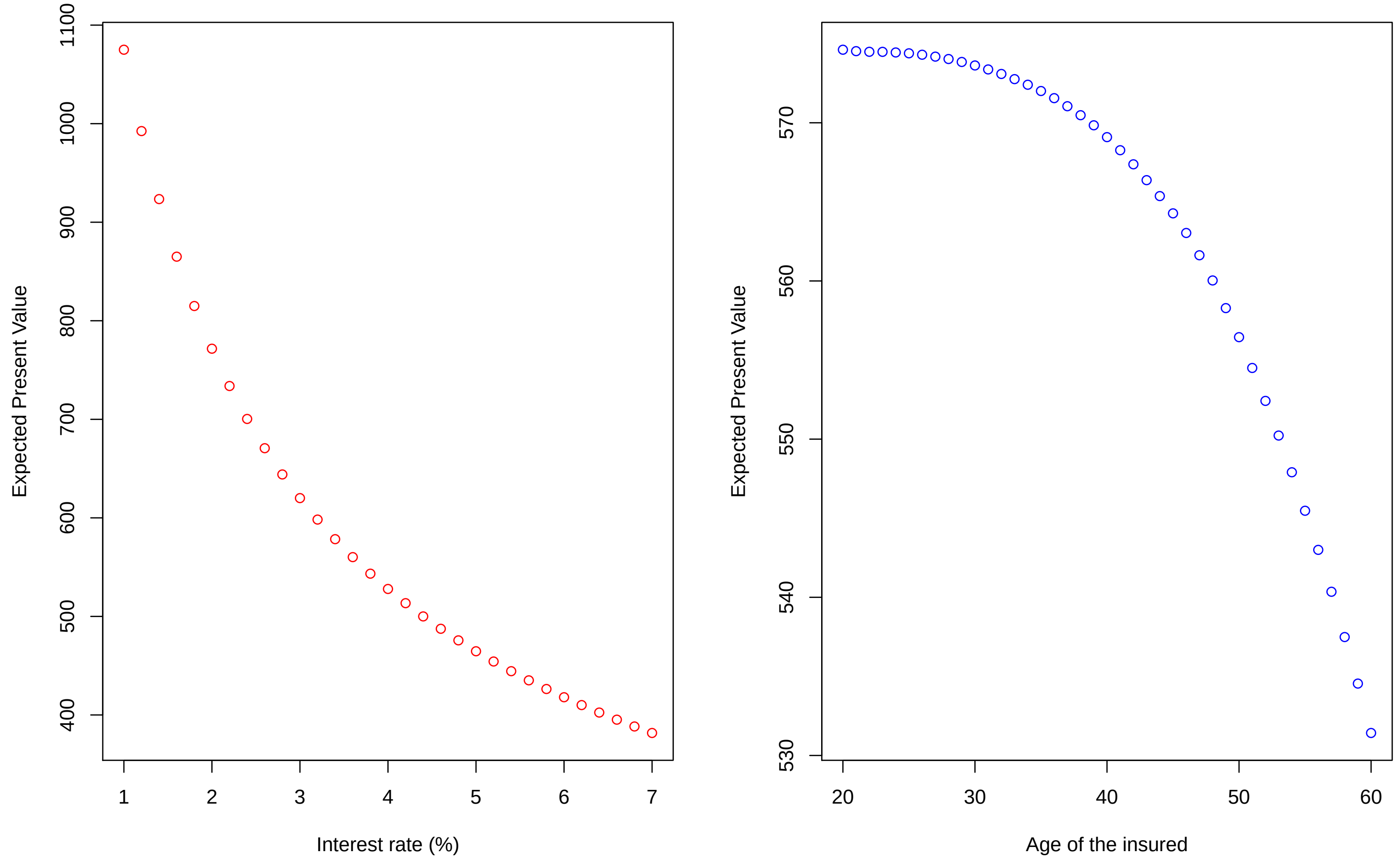
```
> LxTD<-TD$Lx
>  TLAI <- function(capital=1,m=1,n,Lx=TD$Lx,age,rate=.03)
```

```
+  {
+      proba <- Lx[age+1+m:n]/Lx[age+1]
+      vap <- sum((1/(1+rate)^(m:n))*proba*capital)
+      return(vap)
+  }
>  TLAI(capital=100,n=20,age=40)
[1] 1417.045
```

It is possible to visualize the impact of the discount factor i and the age x on that expected present value

```
> TLAI.R <- function(T){TLAI(capital=100,n=20,age=40,rate=T)}
> vect.TLAI.R <- Vectorize(TLAI.R)
> RT <- seq(.01,.07,by=.002)
> TLAI.A <- function(A){VAP(capital=100,n=20,age=A,rate=.035)}
> vect.TLAI.A <- Vectorize(TLAI.A)
> AGE <- seq(20,60)
> par(mfrow = c(1, 2))
> plot(100*RT,vect.TLAI.R(TAUX),xlab="discount rate (%)",
+ ylab="Expected Presebt Value")
```

```
> plot(AGE,vect.TLAI.A(AGE),xlab="age of insured",
+ ylab="Expected Present Value")
```



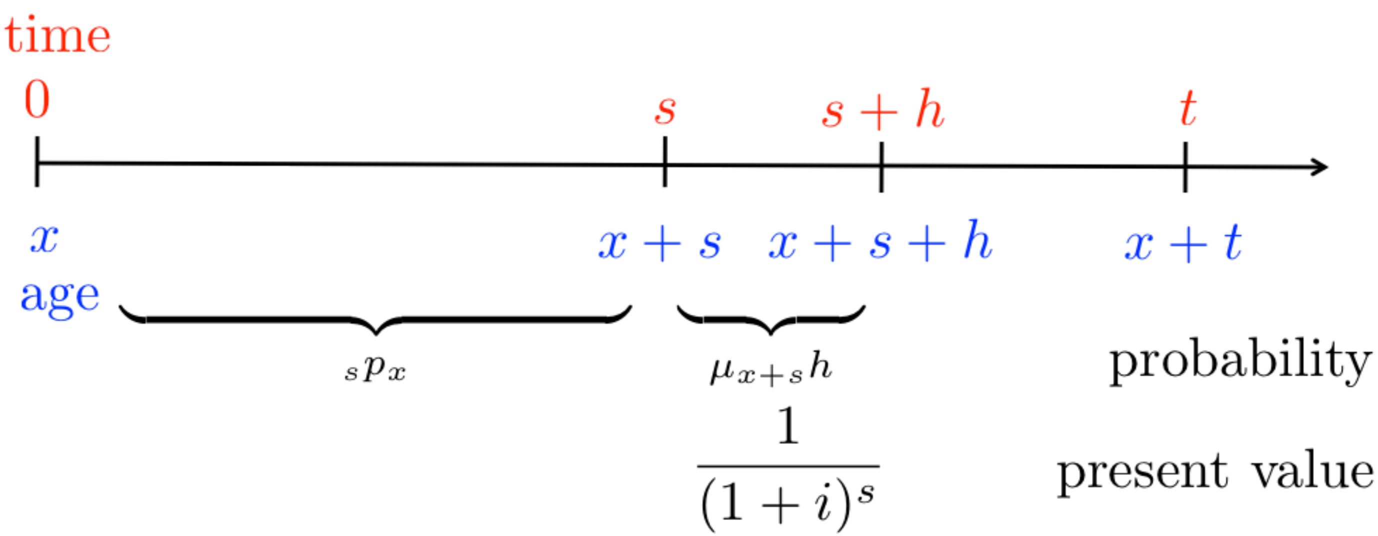
Whole life insurance, continuous case

For life (x), the present value of a benefit of \$1 payable immediately at death is

$$Z = \nu^{T_x} = (1+i)^{-T_x}$$

The expected present value (or actuarial value),

$$\overline{A}_x = \mathbb{E}(\nu^{T_x}) = \int_0^\infty (1+i)^t \cdot {}_tp_x \cdot \mu_{x+t} dt$$



Whole life insurance, annual case

For life (x), present value of a benefit of \$1 payable at the end of the year of death

$$Z = \nu^{\lfloor T_x \rfloor + 1} = (1+i)^{-\lfloor T_x \rfloor + 1}$$

The expected present value (or actuarial value),

$$A_x = \mathbb{E}(\nu^{\lfloor T_x \rfloor + 1}) = \sum_{k=0}^{\infty} (1+i)^{k+1} \cdot {}_{k|}q_x$$

Remark : recursive formula

$$A_x = \nu \cdot q_x + \nu \cdot p_x \cdot A_{x+1}.$$

Term insurance, continuous case

For life (x), present value of a benefit of \$1 payable immediately at death, if death occurs within a fixed term n

$$Z = \begin{cases} \nu^{T_x} = (1+i)^{-T_x} \text{ if } T_x \leq n \\ 0 \text{ if } T_x > n \end{cases}$$

The expected present value (or actuarial value),

$$\overline{A}^{1}_{x:\overline{n}|} = \mathbb{E}(Z) = \int_0^n (1+i)^t \cdot {}_tp_x \cdot \mu_{x+t} dt$$

Term insurance, discrete case

For life (x), present value of a benefit of \$1 payable at the end of the year of death, if death occurs within a fixed term n

$$Z = \begin{cases} \nu^{\lfloor T_x \rfloor + 1} = (1+i)^{-(\lfloor T_x \rfloor + 1)} \text{ if } \lfloor T_x \rfloor \leq n-1 \\ 0 \text{ if } \lfloor T_x \rfloor \geq n \end{cases}$$

The expected present value (or actuarial value),

$$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \nu^{k+1} \cdot {}_{k|}q_x$$

It is possible to define a matrix $\boldsymbol{A} = [A^{1}_{x:\overline{n}|}]$ using

```
> A<- matrix(NA,m,m-1)
> for(j in 1:(m-1)){ A[,j]<-cumsum(1/(1+i)^(1:m)*d[,j]) }
> Ax <- A[nrow(A),1:(m-2)]
```

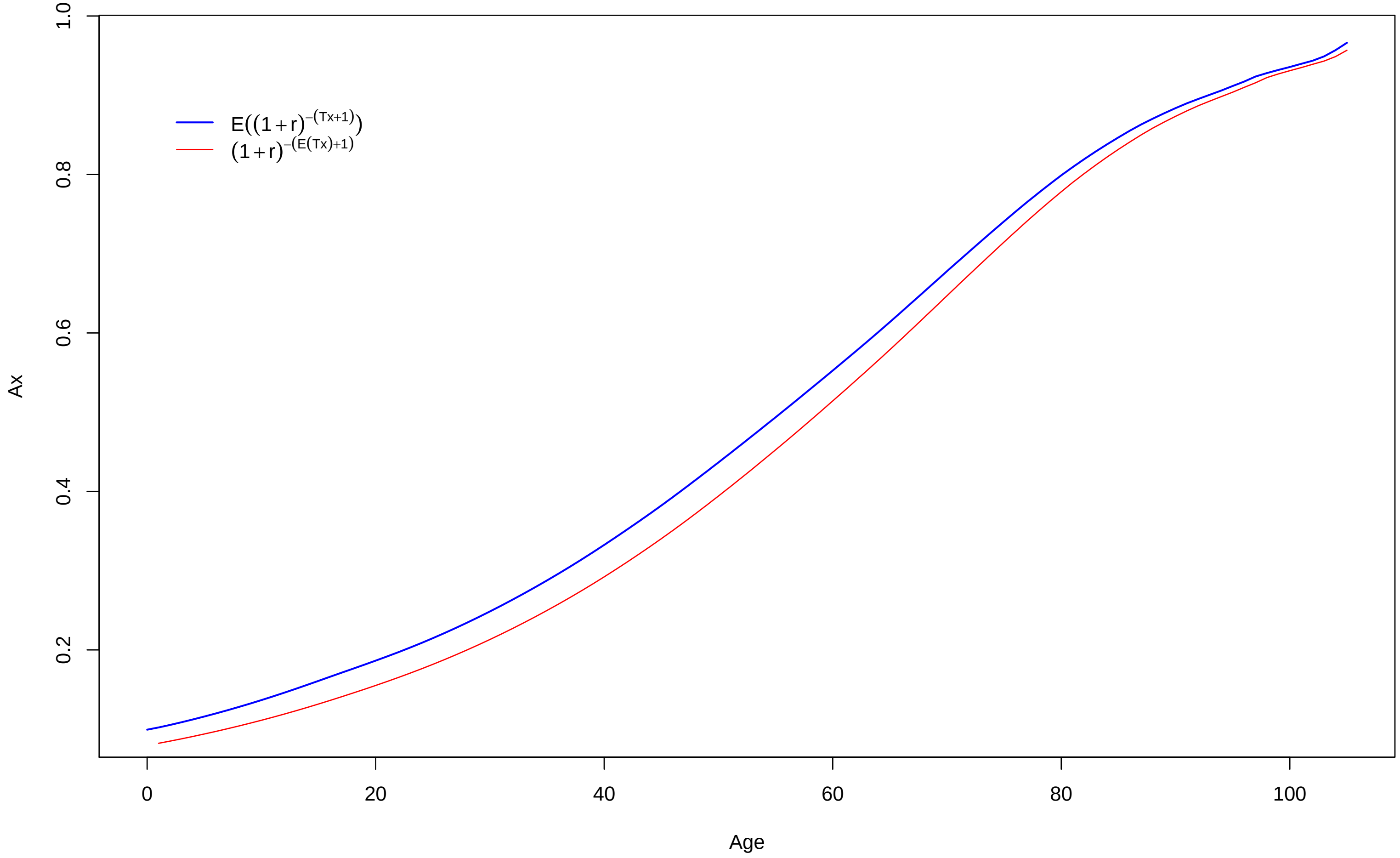
Term insurance, discrete case

Remark : recursion formula

$$A^{1}_{x:\overline{n}|} = \nu \cdot q_x + \nu \cdot p_x \cdot A^{1}_{x:\overline{n}|}.$$

Note that it is possible to compare $\mathbb{E}(\nu^{\lfloor T_x \rfloor + 1})$ and $\nu^{\mathbb{E}(\lfloor T_x \rfloor)+1}$

```
> EV <- Vectorize(esp.vie)
> plot(0:105,Ax,type="l",xlab="Age",lwd=1.5)
> lines(1:105,v^(1+EV(1:105)),col="grey")
> legend(1,.9,c(expression(E((1+r)^-(Tx+1))),expression((1+r)^-(E(Tx)+1))),
+ lty=1,col=c("black","grey"),lwd=c(1.5,1),bty="n")
```


$E((1+r)^{-(Tx+1)})$
$(1+r)^{-(E(Tx)+1)}$
Ax
1.0
0.8
0.6
0.4
0.2
0
20
40
60
80
100
Age

Pure endowment

A pure endowment benefit of \$1, issued to a life aged x, with term of n years has present value

$$Z = \begin{cases} 0 \text{ if } T_x < n \\ \nu^n = (1+i)^{-n} \text{ if } T_x \geq n \end{cases}$$

The expected present value (or actuarial value),

$$A_{x:\overline{n}|}^{\;\;1} = \nu^n \cdot {}_np_x$$

```
> E <- matrix(0,m,m)
> for(j in 1:m){ E[,j] <- (1/(1+i)^(1:m))*p[,j] }
> E[10,45]
[1] 0.663491
> p[10,45]/(1+i)^10
[1] 0.663491
```

Endowment insurance

A pure endowment benefit of \$1, issued to a life aged x, with term of n years has present value

$$Z = \nu^{\min\{T_x,n\}} = \begin{cases} \nu^{T_x} = (1+i)^{-T_x} \text{ if } T_x < n \\ \nu^n = (1+i)^{-n} \text{ if } T_x \geq n \end{cases}$$

The expected present value (or actuarial value),

$$\overline{A}_{x:\overline{n}|} = \overline{A}^{\,1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$$

Discrete endowment insurance

A pure endowment benefit of \$1, issued to a life aged x, with term of n years has present value

$$Z = \nu^{\min\{\lfloor T_x \rfloor + 1, n\}} = \begin{cases} \nu^{\lfloor T_x \rfloor + 1} \text{ if } \lfloor T_x \rfloor \leq n \\ \nu^{n} \text{ if } \lfloor T_x \rfloor \geq n \end{cases}$$

The expected present value (or actuarial value),

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$$

Remark : recursive formula

$$A_{x:\overline{n}|} = \nu \cdot q_x + \nu \cdot p_x \cdot A_{x+1:\overline{n-1}|}.$$

Deferred insurance benefits

A benefit of \$1, issued to a life aged x, provided that (x) dies between ages $x+u$ and $x+u+n$ has present value

$$Z = \nu^{\min\{T_x,n\}} = \begin{cases} \nu^{T_x} = (1+i)^{-T_x} \text{ if } u \leq T_x < u+n \\ 0 \text{ if } T_x < u \text{ or } T_x \geq u+n \end{cases}$$

The expected present value (or actuarial value),

$${}_{u|}\overline{A}^{\,1}_{x:\overline{n}|} = \mathbb{E}(Z) = \int_u^{u+n} (1+i)^t \cdot {}_tp_x \cdot \mu_{x+t} dt$$

Annuities

An *annuity* is a series of payments that might depend on

- the timing payment
  - beginning of year : annuity-due
  - end of year : annuity-immediate
- the maturity (n)
- the frequency of payments (more than once a year, even continuously)
- benefits

Annuities certain

For integer n, consider an annuity (certain) of \$1 payable annually in advance for n years. Its present value is

$$\ddot{a}_{\overline{n|}} = \sum_{k=0}^{n-1} \nu^k = 1 + \nu + \nu^2 + \cdots + \nu^{n-1} = \frac{1-\nu^n}{1-\nu} = \frac{1-\nu^n}{d}$$

In the case of a payment in arrear for n years,

$$a_{\overline{n|}} = \sum_{k=1}^{n} \nu^k = \nu + \nu^2 + \cdots + \nu^{n-1} + \nu^n = \ddot{a}_{\overline{n|}} + (\nu^n - 1) = \frac{1-\nu^n}{i}.$$

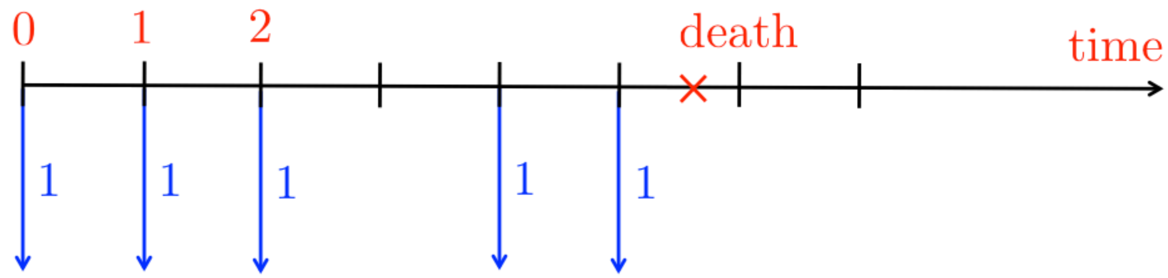
Note that it is possible to consider a continuous version

$$\overline{a}_{\overline{n|}} = \int_0^n \nu^t dt = \frac{\nu^n - 1}{\log(\nu)}$$

Whole life annuity-due

Annuity of \$1 per year, payable annually in advance throughout the lifetime of an individual aged x,

$$Z = \sum_{k=0}^{\lfloor T_x \rfloor} \nu^k = 1 + \nu + \nu^2 + \cdots + \nu^{\lfloor T_x \rfloor} = \frac{1 - \nu^{1+\lfloor T_x \rfloor}}{1 - \nu} = \ddot{a}_{\overline{\lfloor T_x \rfloor + 1}|}$$



Whole life annuity-due

The expected present value (or actuarial value),

$$\ddot{a}_x = \mathbb{E}(Z) = \frac{1 - \mathbb{E}\left(\nu^{1+\lfloor T_x \rfloor}\right)}{1-\nu} = \frac{1 - A_x}{1-\nu}$$

thus,

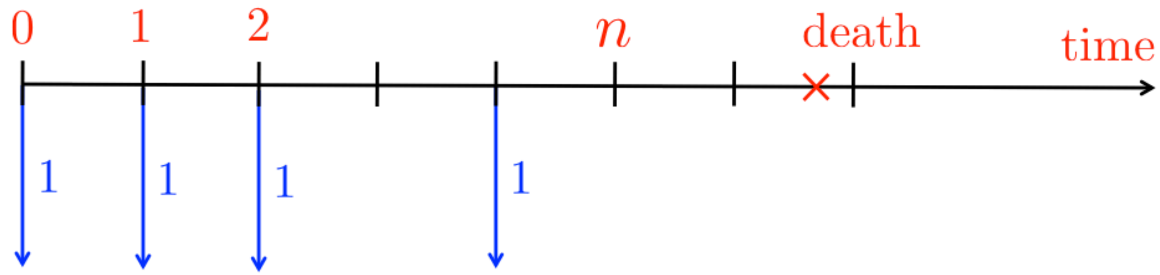
$$\ddot{a}_x = \sum_{k=0}^{\infty} \nu^k \cdot {}_kp_x = \sum_{k=0}^{\infty} {}_kE_x = \frac{1-A_x}{1-\nu}$$

(or conversely $A_x = 1 - [1-\nu](1-\ddot{a}_x)$).

Temporary life annuity-due

Annuity of \$1 per year, payable annually in advance, at times $k = 0, 1, \cdots, n-1$ provided that (x) survived to age $x + k$

$$Z = \sum_{k=0}^{\min\{\lfloor T_x \rfloor, n\}} \nu^k = 1 + \nu + \nu^2 + \cdots + \nu^{\min\{\lfloor T_x \rfloor, n\}} = \frac{1 - \nu^{1+\min\{\lfloor T_x \rfloor, n\}}}{1 - \nu}$$



Temporary life annuity-due

The expected present value (or actuarial value),

$$\ddot{a}_{x:\overline{n}|} = \mathbb{E}(Z) = \frac{1 - \mathbb{E}\left(\nu^{1+\min\{\lfloor T_x \rfloor, n\}}\right)}{1-\nu} = \frac{1 - A_{x:\overline{n}|}}{1-\nu}$$

thus,

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \nu^k \cdot {}_kp_x = \frac{1 - A_{x:\overline{n}|}}{1-\nu}$$

The code to compute matrix $\ddot{\boldsymbol{A}} = [\ddot{a}_{x:\overline{n}|}]$ is

```
> adot<-matrix(0,m,m)
> for(j in 1:(m-1)){ adot[,j]<-cumsum(1/(1+i)^(0:(m-1))*c(1,p[1:(m-1),j])) }
> adot[nrow(adot),1:5]
[1] 26.63507 26.55159 26.45845 26.35828 26.25351
```

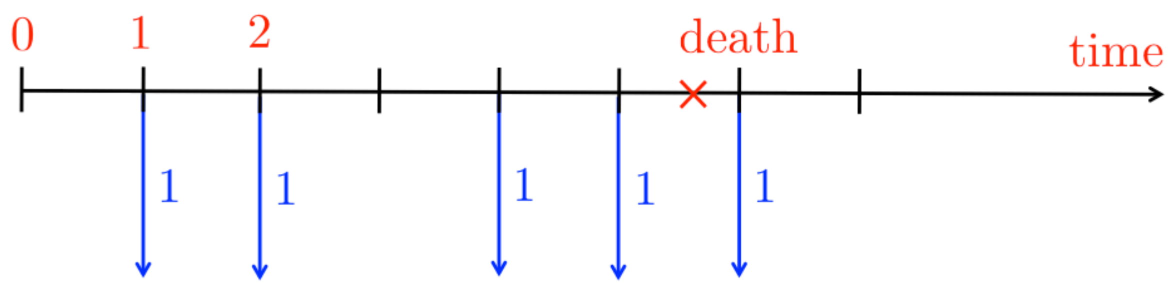
Whole life immediate annuity

Annuity of \$1 per year, payable annually in arrear, at times $k = 1, 2, \cdots$, provided that (x) survived

$$Z = \sum_{k=1}^{\lfloor T_x \rfloor} \nu^k = \nu + \nu^2 + \cdots + \nu^{\lfloor T_x \rfloor}$$

The expected present value (or actuarial value),

$$a_x = \mathbb{E}(Z) = \ddot{a}_x - 1.$$



Term immediate annuity

Annuity of \$1 per year, payable annually in arrear, at times $k = 1, 2, \cdots, n$ provided that (x) survived

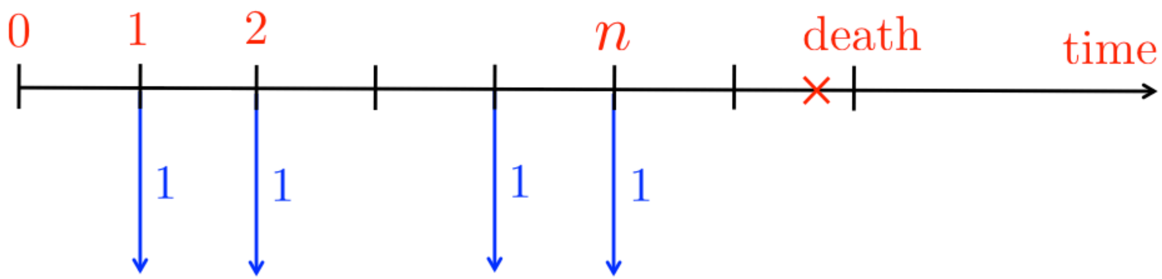
$$Z = \sum_{k=1}^{\min\{\lfloor T_x \rfloor, n\}} \nu^k = \nu + \nu^2 + \cdots + \nu^{\min\{\lfloor T_x \rfloor, n\}}.$$

The expected present value (or actuarial value),

$$a_{x:\overline{n}|} = \mathbb{E}(Z) = \sum_{k=1}^{n} \nu^k \cdot {}_k p_x$$

thus,

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + \nu^n \cdot {}_n p_x$$



Whole and term continuous annuities

Those relationships can be extended to the case where annuity is payable continuously, at rate of \$1 per year, as long as (x) survives.

$$\overline{a}_x = \mathbb{E}\left(\frac{\nu^{T_x} - 1}{\log(\nu)}\right) = \int_0^\infty e^{-\delta t} \cdot {}_t p_x dt$$

where $\delta = -\log(\nu)$.

It is possible to consider also a term continuous annuity

$$\overline{a}_{x:\overline{n}|} = \mathbb{E}\left(\frac{\nu^{\min\{T_x,n\}} - 1}{\log(\nu)}\right) = \int_0^n e^{-\delta t} \cdot {}_t p_x dt$$

Deferred annuities

It is possible to pay a benefit of \$1 at the beginning of each year while insured (x) survives from $x+h$ onward. The expected present value is

$$ {}_{h|}\ddot{a}_x = \sum_{k=h}^{\infty} \frac{1}{(1+i)^k} \cdot {}_k p_x = \sum_{k=h}^{\infty} {}_k E_x = \ddot{a}_x - \ddot{a}_{x:\overline{h}|} $$

One can consider deferred temporary annuities

$$ {}_{h|n}\ddot{a}_x = \sum_{k=h}^{h+n-1} \frac{1}{(1+i)^k} \cdot {}_k p_x = \sum_{k=h}^{h+n-1} {}_k E_x . $$

Remark : again, recursive formulas can be derived

$$ \ddot{a}_x = \ddot{a}_{x:\overline{h}|} + {}_{h|}\ddot{a}_x \text{ for all } h \in \mathbb{N}_* . $$

Deferred annuities

With h fixed, it is possible to compute matrix $\ddot{\boldsymbol{A}}_h = [{}_{h|n}\ddot{a}_x]$

```
> h <- 1
> adoth <- matrix(0,m,m-h)
> for(j in 1:(m-1-h)){ adoth[,j]<-cumsum(1/(1+i)^(h+0:(m-1))*p[h+0:(m-1),j]) }
> adoth[nrow(adoth),1:5]
[1] 25.63507 25.55159 25.45845 25.35828 25.25351
```

Joint life and last survivor probabilities

It is possible to consider life insurance contracts on two individuals, (x) and (y), with remaining lifetimes T_x and T_y respectively. Their joint cumulative distribution function is $F_{x,y}$ while their joint survival function will be $\overline{F}_{x,y}$, where

$$\begin{cases} F_{x,y}(s,t) = \mathbb{P}(T_x \leq s, T_y \leq t) \\ \overline{F}_{x,y}(s,t) = \mathbb{P}(T_x > s, T_y > t) \end{cases}$$

Define the joint life status, (xy), with remaining lifetime $T_{xy} = \min\{T_x, T_y\}$ and let

$${}_tq_{xy} = \mathbb{P}(T_{xy} \leq t) = 1 - {}_tp_{xy}$$

Define the last-survivor status, $(\overline{xy})$, with remaining lifetime $T_{\overline{xy}} = \max\{T_x, T_y\}$ and let

$${}_tq_{\overline{xy}} = \mathbb{P}(T_{\overline{xy}} \leq t) = 1 - {}_tp_{\overline{xy}}$$

Joint life and last survivor probabilities

Assuming independence

$$_hp_{xy} = {}_hp_x \cdot {}_hp_y,$$

while

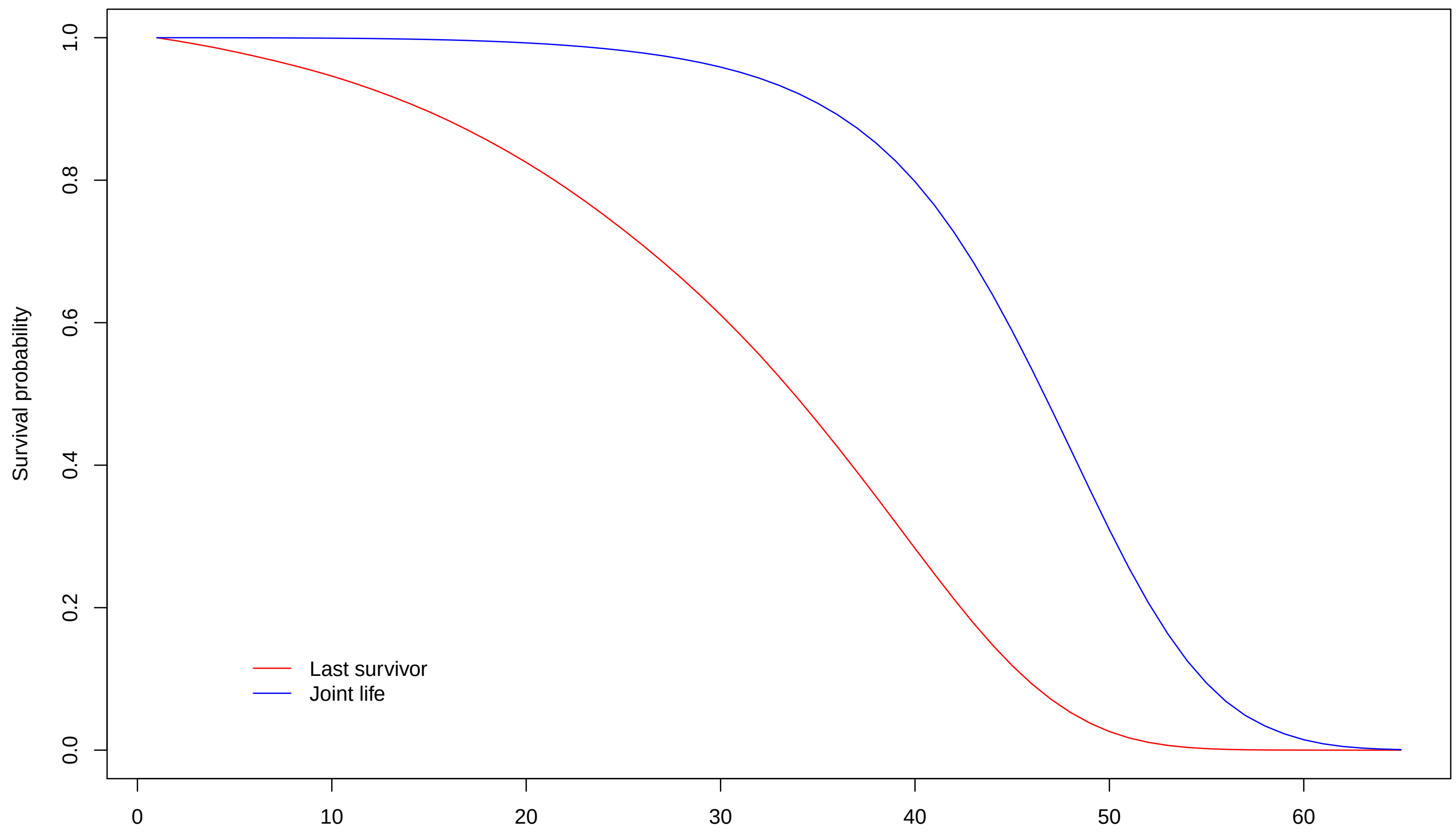
$$_hp_{\overline{xy}} = {}_hp_x + {}_hp_y - {}_hp_{xy}.$$

```
> pxt=function(T,a,h){ T$Lx[T$Age==a+h]/T$Lx[T$Age==a] }
> pxt(TD8890,40,10)*pxt(TV8890,42,10)
[1] 0.9376339
> pxytjoint=function(Tx,Ty,ax,ay,h){ pxt(Tx,ax,h)*pxt(Ty,ay,h) }
> pxytjoint(TD8890,TV8890,40,42,10)
[1] 0.9376339
> pxytlastsurv=function(Tx,Ty,ax,ay,h){ pxt(Tx,ax,h)*pxt(Ty,ay,h) -
+  pxytjoint(Tx,Ty,ax,ay,h)}
> pxytlastsurv(TD8890,TV8890,40,42,10)
[1] 0.9991045
```

Joint life and last survivor probabilities

It is possible to plot

```
> JOINT=rep(NA,65)
> LAST=rep(NA,65)
> for(t in 1:65){
+ JOINT[t]=pxytjoint(TD8890,TV8890,40,42,t-1)
+ LAST[t]=pxytlastsurv(TD8890,TV8890,40,42,t-1) }
> plot(1:65,JOINT,type="l",col="grey",xlab="",ylab="Survival probability")
> lines(1:65,LAST)
> legend(5,.15,c("Dernier survivant","Vie jointe"),lty=1, col=c("black","grey"),bty="n")
```


Survival probability
0.0
0.2
0.4
0.6
0.8
1.0
0
10
20
30
40
50
60
Last survivor
Joint life

Joint life and last survivor insurance benefits

For a joint life status (xy), consider a whole life insurance providing benefits at the first death. Its expected present value is

$$A_{xy} = \sum_{k=0}^{\infty} \nu^k \cdot {}_{k|}q_{xy}$$

For a last-survivor status $(\overline{xy})$, consider a whole life insurance providing benefits at the last death. Its expected present value is

$$A_{\overline{xy}} = \sum_{k=0}^{\infty} \nu^k \cdot {}_{k|}q_{\overline{xy}} = \sum_{k=0}^{\infty} \nu^k \cdot [{}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}]$$

Remark : Note that $A_{xy} + A_{\overline{xy}} = A_x + A_y$.

Joint life and last survivor insurance benefits

For a joint life status (xy), consider a whole life insurance providing annuity at the first death. Its expected present value is

$$\ddot{a}_{xy} = \sum_{k=0}^{\infty} \nu^k \cdot {}_kp_{xy}$$

For a last-survivor status $(\overline{xy})$, consider a whole life insurance providing annuity at the last death. Its expected present value is

$$\ddot{a}_{\overline{xy}} = \sum_{k=0}^{\infty} \nu^k \cdot {}_kp_{\overline{xy}}$$

Remark : Note that $\ddot{a}_{xy} + \ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y$.

Reversionary insurance benefits

A reversionary annuity commences upon the death of a specified status (say (y)) if a second (say (x)) is alive, and continues thereafter, so long as status (x) remains alive. Hence, reversionary annuity to (x) after (y) is

$$a_{y|x} = \sum_{k=1}^{\infty} \nu^k \cdot {}_kp_x \cdot {}_kq_y = \sum_{k=1}^{\infty} \nu^k \cdot {}_kp_x \cdot [1 - {}_kp_y] = a_x - a_{xy}.$$

Premium calculation

Fundamental theorem : (equivalence principle) at time $t = 0$,

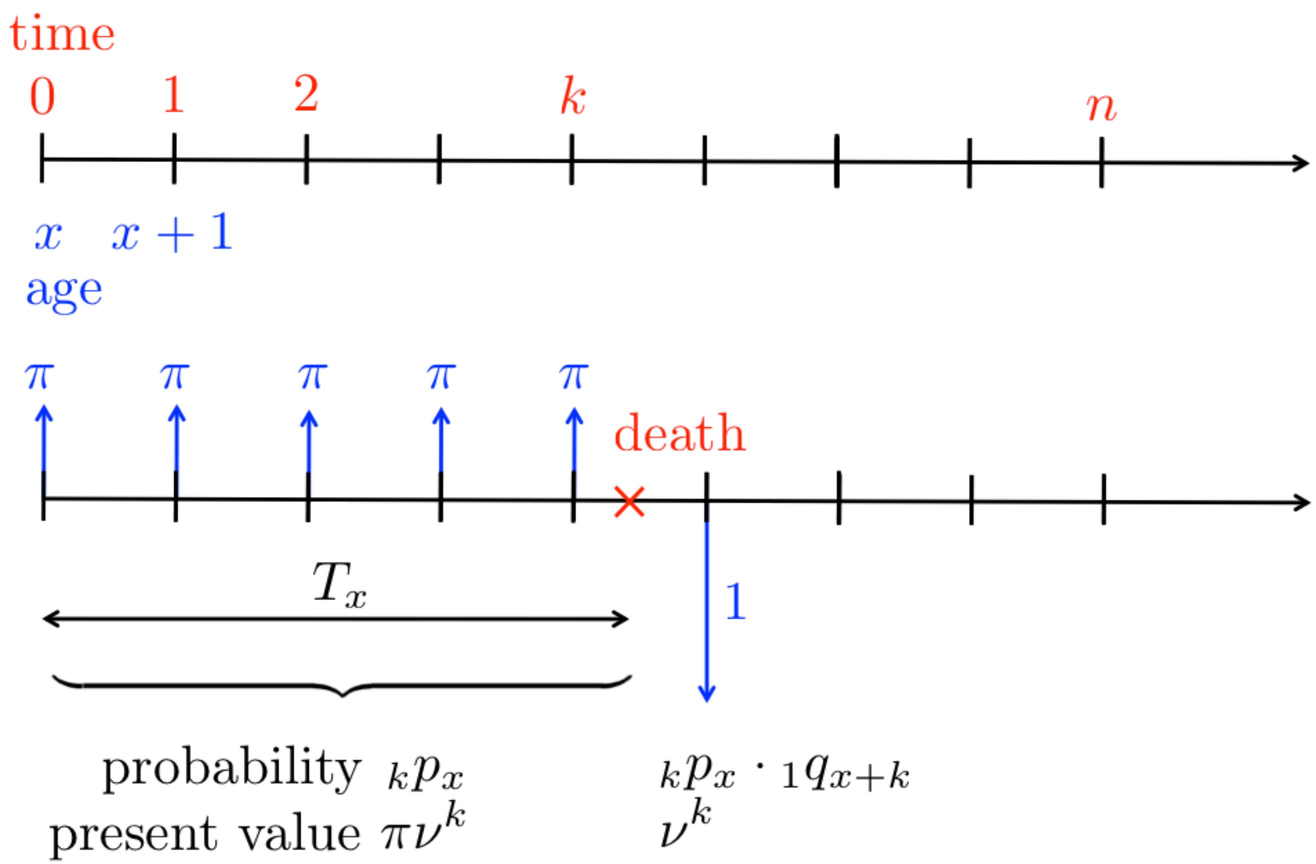
$$\mathbb{E}(\text{present value of net premium income}) = \mathbb{E}(\text{present value of benefit outgo})$$

Let

$$L_0 = \text{present value of future benefits - present value of future net premium}$$

Then $\mathbb{E}(L_0) = 0$.

Example : consider a n year endowment policy, paying C at the end of the year of death, or at maturity, issues to (x). Premium P is paid at the beginning of year year throughout policy term. Then, if $K_n = \min\{K_x + 1, n\}$


time
0
1
2
k
n
x
$x+1$
age
π
π
π
π
π
death
T_x
1
probability ${}_kp_x$
present value $\pi\nu^k$
${}_kp_x \cdot {}_1q_{x+k}$
ν^k

Premium calculation

$$L_0 = \underbrace{C \cdot \nu^{K_n}}_{\text{future benefit}} - \underbrace{P \cdot \ddot{a}_{\overline{K_n}|}}_{\text{net premium}}$$

Thus,

$$\mathbb{E}(L_0) = C \cdot A_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|} = 0, \text{ thus } P = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

```
>  x <-50; n <-30
>  premium <-A[n,x]/adot[n,x]
>  sum(premium/(1+i)^(0:(n-1))*c(1,p[1:(n-1),x]))
[1] 0.3047564
>  sum(1/(1+i)^(1:n)*d[1:n,x])
[1] 0.3047564
```

Policy values

From year k to year $k+1$, the profit (or loss) earned during that period depends on interest and mortality (cf. Thiele's differential equation).

For convenience, let $EPV^t_{[t_1,t_2]}$ denote the *expected present value*, calculated at time t of benefits or premiums over period $[t_1, t_2]$. Then

$$\underbrace{EPV^0_{[0,n]}(\text{benefits})}_{\text{insurer}} = \underbrace{EPV^0_{[0,n]}(\text{net premium})}_{\text{insured}}$$

for a contact that ends at after n years.

Remark : Note that $EPV^0_{[k,n]} = EPV^k_{[k,n]} \cdot {}_kE_x$ where

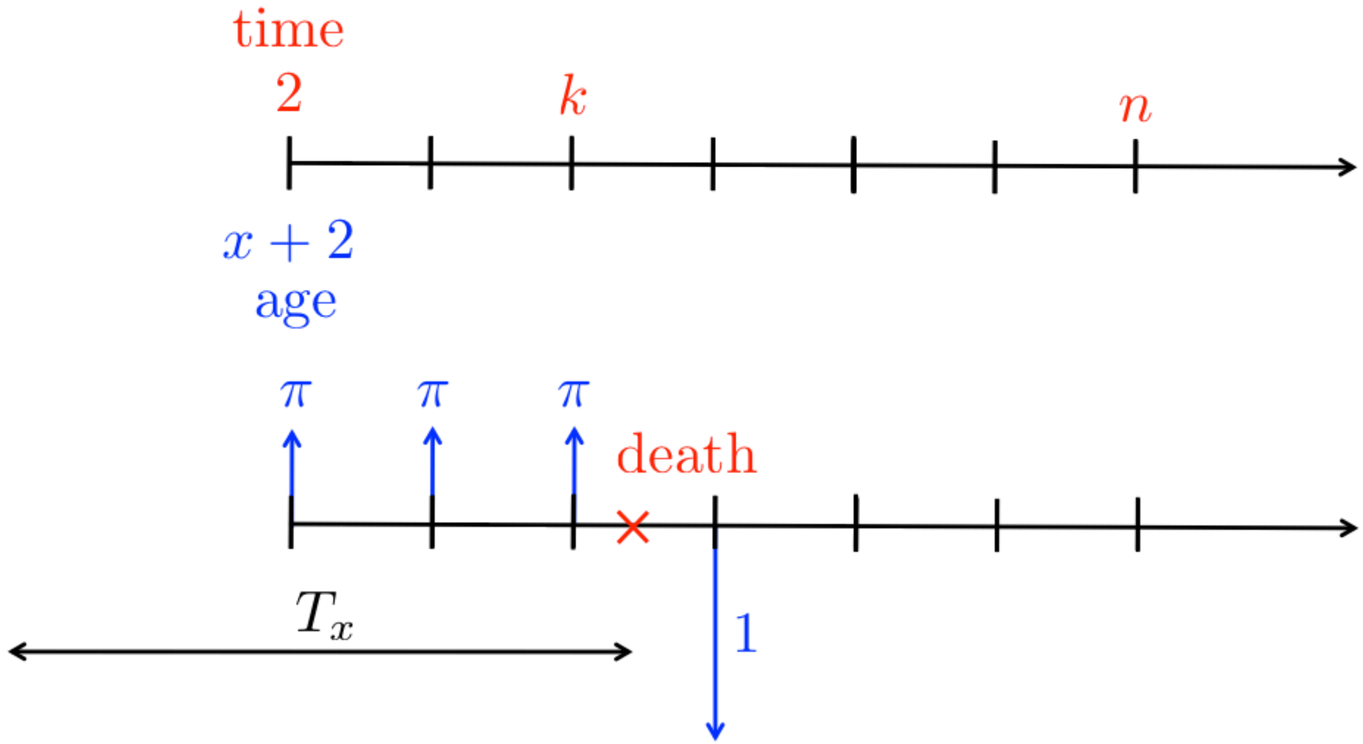
$${}_kE_x = \frac{1}{(1+i)^k} \cdot \mathbb{P}(T_x > k) = \nu^k \cdot {}_kp_x$$

Policy values and reserves

Define

$$L_t = \text{present value of future benefits - present value of future net premium}$$

where present values are calculated at time t.



For convenient, let $EPV^t_{(t_1,t_2]}$ denote the *expected present value*, calculated at time t of benefits or premiums over period $(t_1, t_2]$. Then

$$\mathbb{E}_k(L_k) = \underbrace{EPV^k_{(k,n]}(\text{benefits})}_{\text{insurer}} - \underbrace{EPV^0_{(k,n]}(\text{net premium})}_{\text{insurer}} = {}_kV(k).$$

Example : consider a n year endowment policy, paying C at the end of the year of death, or at maturity, issues to (x). Premium P is paid at the beginning of year year throughout policy term. Let $k \in \{0, 1, 2, \cdots, n-1, n\}$. From that prospective relationship

$$_kV(k) = {}_{n-k}A_{x+k} - \pi \cdot {}_{n-k}\ddot{a}_{x+k}$$

```
> VP <- diag(A[n-(0:(n-1)),x+(0:(n-1))])-
+ primediag(adot[n-(0:(n-1)),x+(0:(n-1))])
> plot(0:n,c(VP,0),pch=4,xlab="",ylab="Provisions mathématiques",type="b")
```

An alternative is to observe that

$$\mathbb{E}_0(L_k) = \underbrace{EPV^0_{(k,n]}(\text{benefits})}_{\text{insurer}} - \underbrace{EPV^0_{(k,n]}(\text{net premium})}_{\text{insurer}} = {}_kV(0).$$

while

$$\mathbb{E}_0(L_0) = \underbrace{EPV^0_{[0,n]}(\text{benefits})}_{\text{insurer}} - \underbrace{EPV^0_{[0,n]}(\text{net premium})}_{\text{insurer}} = 0.$$

Thus

$$\mathbb{E}_0(L_k) = \underbrace{EPV^0_{[0,k]}(\text{net premium})}_{\text{insurer}} - \underbrace{EPV^0_{[0,k]}(\text{benefits})}_{\text{insurer}} = {}_kV(0).$$

which can be seen as a retrospective relationship.

Here ${}_kV(0) = \pi \cdot {}_k\ddot{a}_x - {}_kA_x$, thus

$${}_kV(k) = \frac{\pi \cdot {}_k\ddot{a}_x - {}_kA_x}{{}_kE_x} = \frac{\pi \cdot {}_k\ddot{a}_x - {}_kA_x}{{}_kE_x}$$

```
> VR <- (premium*adot[1:n,x]-A[1:n,x])/E[1:n,x]
> points(0:n,c(0,VR))
```

Another technique is to consider the variation of the reserve, from $k-1$ to k. This will be the iterative relationship. Here

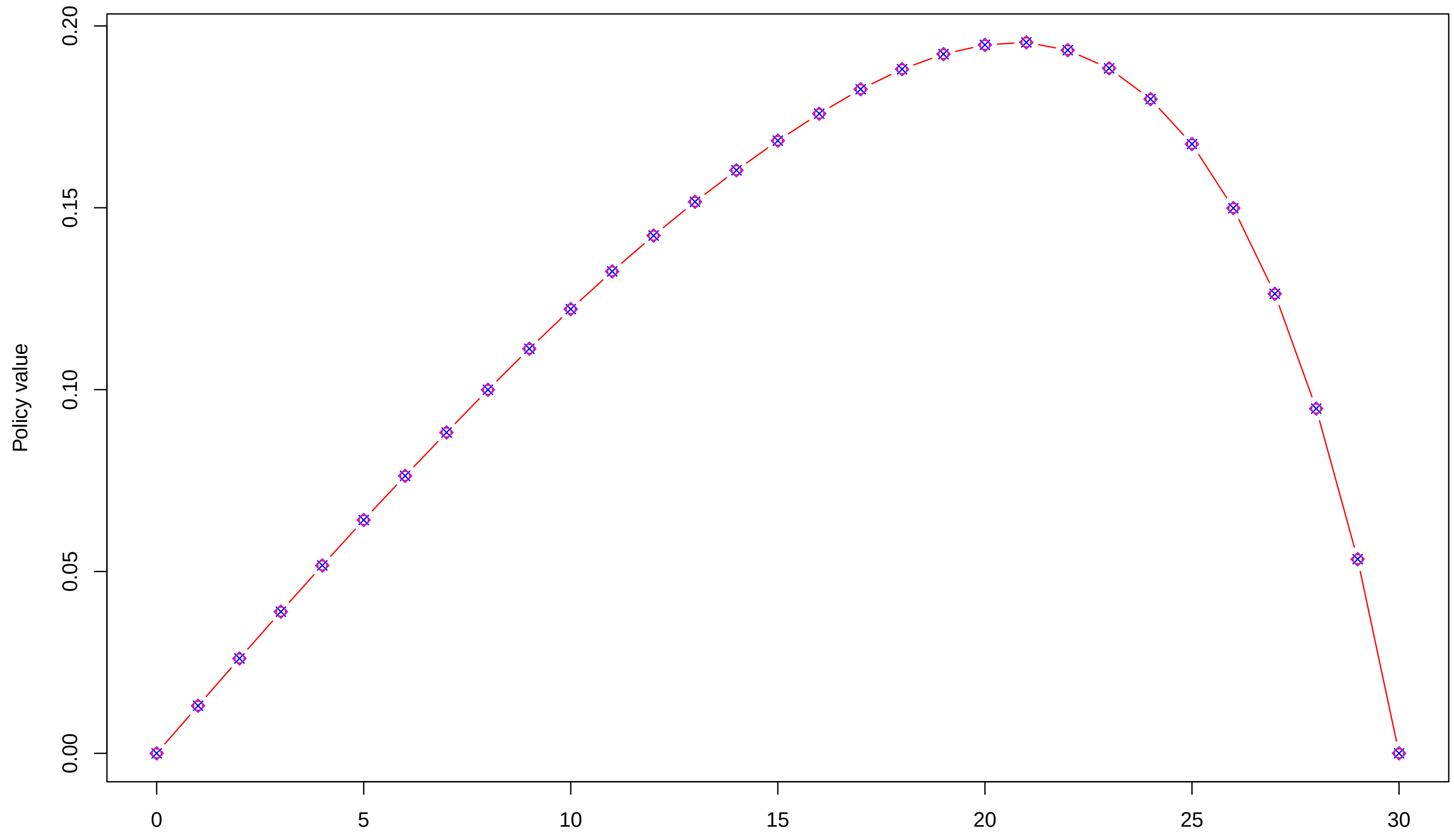
$$_{k}V(k-1) = {}_{k-1}V(k-1) + \pi - {}_{1}A_{x+k-1}.$$

Since $_{k}V(k-1) = {}_{k}V(k) \cdot {}_{1}E_{x+k-1}$ we can derive

$$_{k}V(k) = \frac{{}_{k-1}V_{x}(k-1) + \pi - {}_{1}A_{x+k-1}}{{}_{1}E_{x+k-1}}$$

```
> VI<-0
>  for(k in 1:n){  VI <- c(VI,(VI[k]+prime-A[1,x+k-1])/E[1,x+k-1])  }
>  points(0:n,VI,pch=5)
```

Those three algorithms return the same values, when $x = 50$, $n = 30$ and $i = 3.5\%$



Policy value
0.00
0.05
0.10
0.15
0.20
0
5
10
15
20
25
30

Policy values and reserves : pension

Consider an insured (x), paying a premium over n years, with then a deferred whole life pension (C, yearly), until death. Let m denote the maximum number of years (i.e. $x_{\max} - x$). The annual premium would be

$$\pi = C \cdot \frac{{}_{n|}a_x}{{}_n\ddot{a}_x}$$

Consider matrix ${}_{|}\boldsymbol{A} = [{}_{n|}a_x]$ computed as follows

```
> adiff=matrix(0,m,m)
>  for(i in 1:(m-1)){  adiff[(1+0:(m-i-1)),i] <- E[(1+0:(m-i-1)),i]*a[m,1+i+(0:(m-i-1))] }
```

Yearly pure premium is here the following

```
>  x <- 35
>  n <- 30
>  a[n,x]
[1] 17.31146
>  sum(1/(1+i)^(1:n)*c(p[1:n,x]) )
```

```
[1] 17.31146
> (premium <- adiff[n,x] / (adot[n,x]))
[1] 0.1661761
> sum(1/(1+i)^((n+1):m)*p[(n+1):m,x] )/sum(1/(1+i)^(1:n)*c(p[1:n,x]) )
[1] 0.17311
```

To compute policy values, consider the prospective method, if $k < n$,

$$_kV_x(0) = C \cdot {}_{n-k|}a_{x+k} - {}_{n-k}\ddot{a}_{x+k}.$$

but if $k \geq n$ then

$$_kV_x(0) = C \cdot a_{x+k}.$$

```
> VP <- rep(NA,n-x)
> VP[1:(n-1)] <- diag(adiff[n-(1:(n-1)),x+(1:(n-1))] -
+ adot[n-(1:(n-1)),x+(1:(n-1))]*prime)
> VP[n:(m-x)] <- a[m,x+n:(m-x)]
> plot(x:m,c(0,VP),xlab="Age of the insured",ylab="Policy value")
```

Again, a retrospective method can be used. If $k \leq n$,

$$_kV_x(0) = \frac{\pi \cdot {}_k\ddot{a}_x}{_kE_x}$$

while if $k > n$,

$$_kV_x(0) = \frac{\pi \cdot {}_n\ddot{a}_x - C \cdot {}_{n|k}a_x}{_kE_x}$$

For computations, recall that

$$_{n|k}a_x = \sum_{j=n+1}^{n+k} {}_jE_x = {}_{n|}a_x - {}_{n+k|}a_x$$

It is possible to define a matrix $\boldsymbol{A}_x = [_{n|k}a_x]$ as follows

```
>  adiff[n,x]
[1] 2.996788
>  adiff[min(which(is.na(adiffx[,n])))-1,n]
[1] 2.996788
```

```
> adiff[10,n]
[1] 2.000453
> adiff[n,x]- adiff[n+10,x]
[1] 2.000453
```

The policy values can be computed

```
> VR <- rep(NA,m-x)
> VR[1:(n)] <- adot[1:n,x]*prime/E[1:n,x]
> VR[(n+1):(m-x)] <- (adot[n,x]*prime - (adiff[(n),x]-
+ adiff[(n+1):(m-x),x]) )/E[(n+1):(m-x),x]
> points(x:m,c(0,VR),pch=4)
```

An finally, an iterative algorithm can be used. If $k \leq n$,

$$ {}_kV_x(0) = \frac{{}_{k-1}V_x(0) + \pi}{{}_1E_{x+k-1}} $$

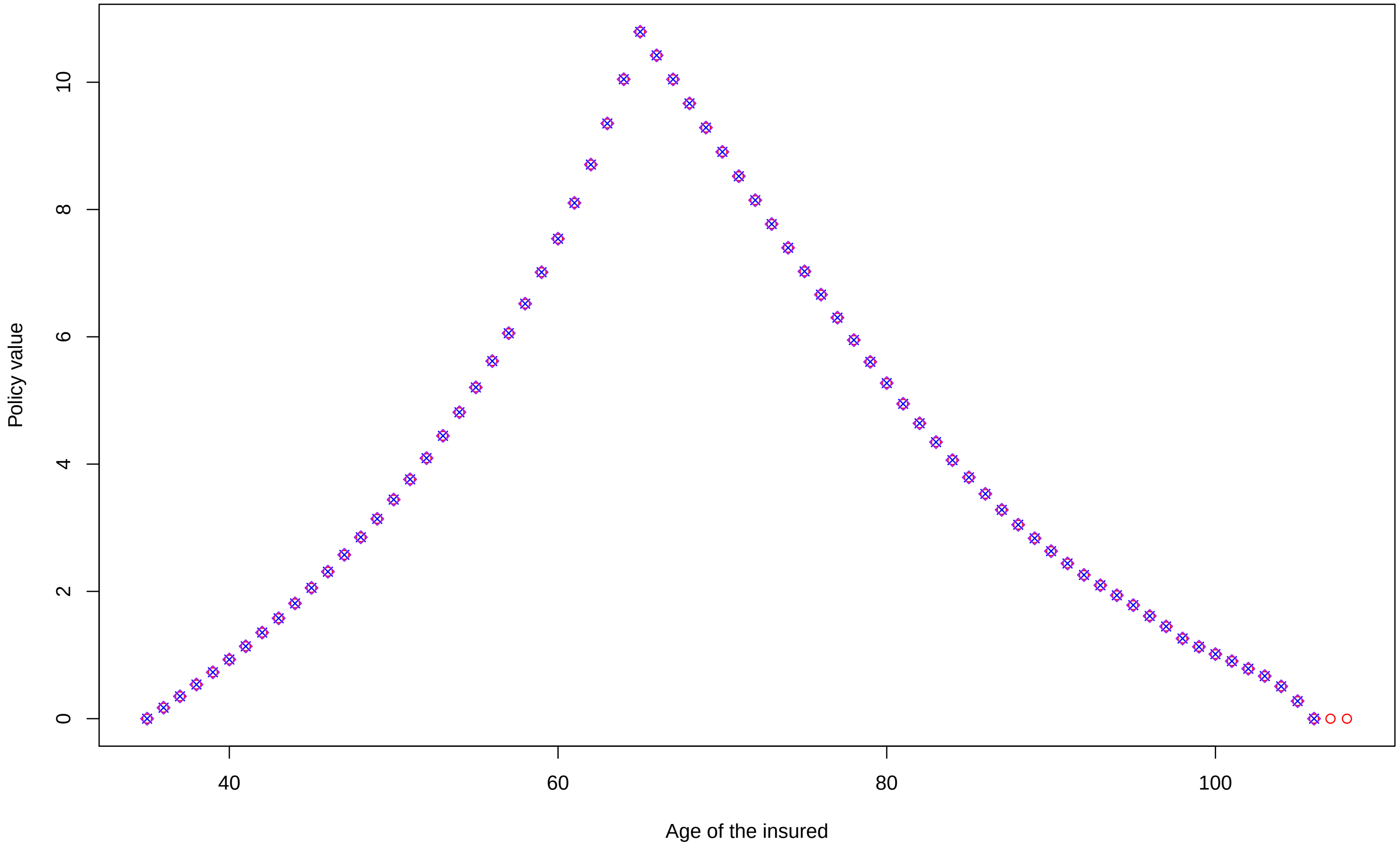
while, if $k > n$

$$ {}_kV_x(0) = \frac{{}_{k-1}V_x(0)}{{}_1E_{x+k-1}} - C. $$

```
>  VI<-0
>  for(k in 1:n){
+  VI<-c(VI,((VI[k]+prime)/E[1,x+k-1]))
+  }
>  for(k in (n+1):(m-x)){
+  VI<-c(VI,((VI[k])/E[1,x+k-1]-1))
+  }
>  points(x:m,VI,pch=5)

> provision<-data.frame(k=0:(m-x),
+ retrospective=c(0,VR),prospective=c(0,VP),
+ iterative=VI)
> head(provision)
  k retrospective prospective iterative
1 0     0.0000000   0.0000000 0.0000000
2 1     0.1723554   0.1723554 0.1723554
3 2     0.3511619   0.3511619 0.3511619
4 3     0.5367154   0.5367154 0.5367154
5 4     0.7293306   0.7293306 0.7293306
6 5     0.9293048   0.9293048 0.9293048
> tail(provision)
```

```
    k retrospective prospective    iterative
69 68     0.6692860   0.6692860 6.692860e-01
70 69     0.5076651   0.5076651 5.076651e-01
71 70     0.2760524   0.2760524 2.760525e-01
72 71     0.0000000   0.0000000 1.501743e-10
73 72           NaN   0.0000000          Inf
74 73           NaN   0.0000000          Inf
```


Policy value
0
2
4
6
8
10
40
60
80
100
Age of the insured

Using recursive formulas

Most quantities in actuarial sciences can be obtained using recursive formulas, e.g.

$$A_x = \mathbb{E}(\nu^{T_x+1}) = \sum_{k=0}^{\infty} \upsilon^{k+1} {}_{k|}q_x = \nu q_x + \nu p_x A_{x+1}$$

or

$$\ddot{a}_x = \sum_{k=0}^{\infty} \nu^k {}_k p_x = 1 + \nu p_x \ddot{a}_{x+1}.$$

Some general algorithms can be used here : consider a sequence $\boldsymbol{u} = (u_n)$ such that

$$u_n = a_n + b_n u_{n+1},$$

where $n = 1, 2, \cdots, m$ assuming that u_{m+1} is known, for some $\boldsymbol{a} = (a_n)$ et

$\boldsymbol{b} = (b_n)$. The general solution is then

$$u_n = \frac{u_{m+1}\prod_{i=0}^{m} b_i + \sum_{j=n}^{m} a_j \prod_{i=0}^{j-1} b_i}{\prod_{i=0}^{n-1} b_i}$$

with convention $b_0 = 1$.

Consider function

```
> recurrence <- function(a,b,ufinal){
+ s <- rev(cumprod(c(1, b)));
+ return(rev(cumsum(s[-1] * rev(a))) + s[1] * ufinal)/rev(s[-1])
+ }
```

For remaining life satifsfies

$$e_x = p_x + p_x \cdot e_{x+1}$$

Le code est alors tout simplement,

```
> Lx <- TD$Lx
> x <- 45
> kpx <- Lx[(x+2):length(Lx)]/Lx[x+1]
> sum(kpx)
[1] 30.32957
> px <- Lx[(x+2):length(Lx)]/Lx[(x+1):(length(Lx)-1)]
> e<- recurrence(px,px,0)
> e[1]
[1] 30.32957
```

For the whole life insurance expected value

$$A_x = \nu q_x + \nu p_x A_{x+1}$$

Here

```
> x <- 20
> qx <- 1-px
> v <- 1/(1+i)
> Ar <- recurrence(a=v*qx,b=v*px,xfinal=v)
```

For instance if $x = 20$,

```
> Ar[1]
[1] 0.1812636
> Ax[20]
[1] 0.1812636
```

An R package for life contingencies ?

Package `lifecontingencies` does (almost) everything we've seen.

From dataset `TD$Lx` define an object of class `lifetable` containing for all ages x survival probabilities p_x, and expected remaining lifetimes e_x.

```
>  TD8890 <- new("lifetable",x=TD$Age,lx=TD$Lx,name="TD8890")
removing NA and 0s
>  TV8890 <- new("lifetable",x=TV$Age,lx=TV$Lx,name="TV8890")
removing NA and 0s
```

An R package for life contingencies ?

```
> TV8890
Life table TV8890

      x     lx        px         ex
1     0 100000 0.9935200 80.2153857
2     1  99352 0.9994162 79.2619494
3     2  99294 0.9996677 78.2881343
4     3  99261 0.9997481 77.3077311
5     4  99236 0.9997783 76.3247626
6     5  99214 0.9997984 75.3400508
7     6  99194 0.9998286 74.3528792
8     7  99177 0.9998387 73.3647956
9     8  99161 0.9998386 72.3765545
10    9  99145 0.9998386 71.3881558
```

That `S4`-class object can be used using standard functions. E.g. ${}_{10}p_{40}$ can be computed through

```
> pxt(TD8890,x=40,t=10)
```

```
[1] 0.9581196
> p[10,40]
[1] 0.9581196
```

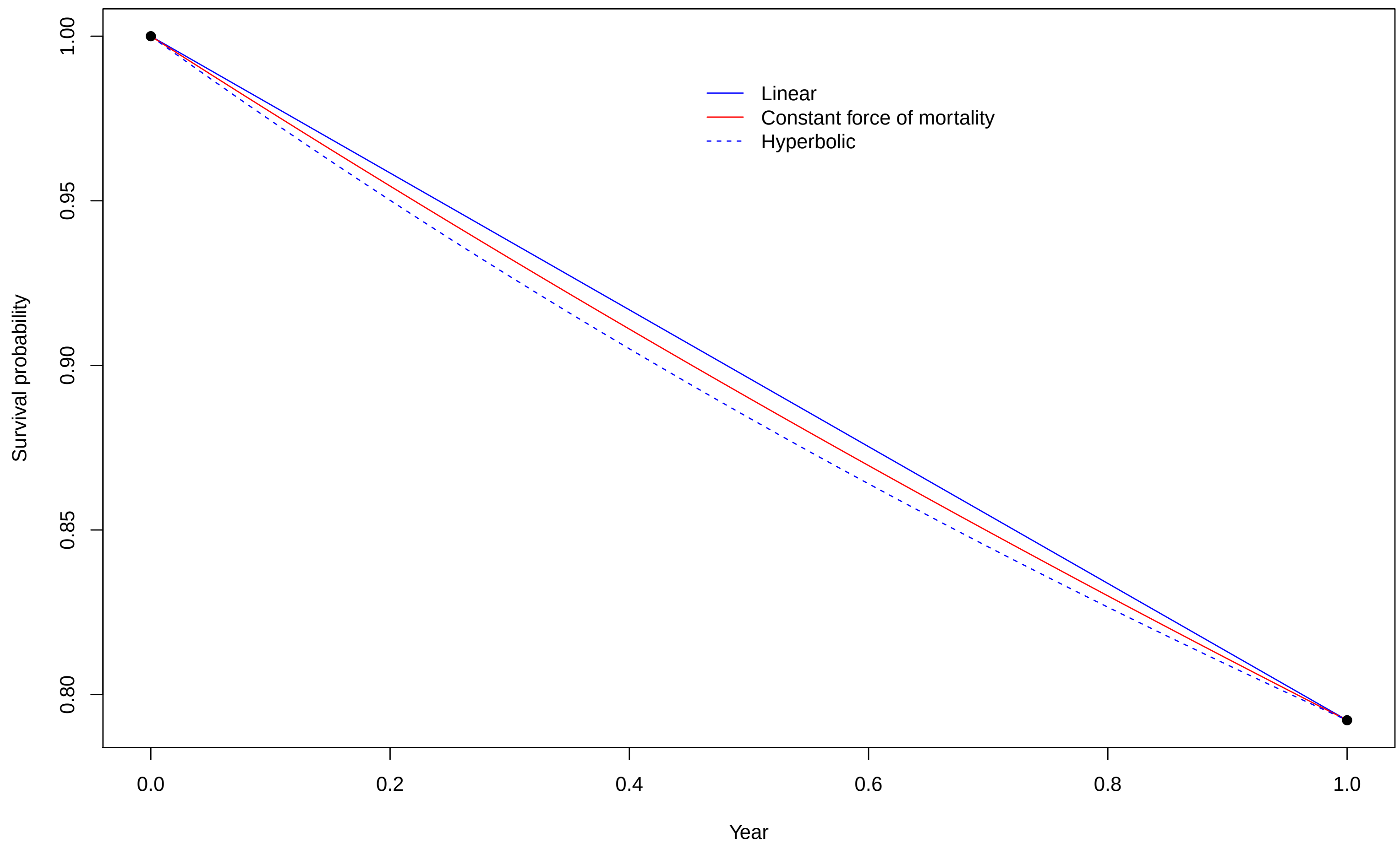
Similarly ${}_{10}q_{40}$, or $\overset{\circ}{e}_{40:\overline{10}|}$ are computed using

```
> qxt(TD8890,40,10)
[1] 0.0418804
> exn(TD8890,40,10)
[1] 9.796076
```

Interpolation of survival probabilities

It is also possible to compute ${}_hp_x$ when h is not necessarily an integer. Linear interpolation, with constant mortality force or hyperbolic can be used

```
> pxt(TD8890,90,.5,"linear")
[1] 0.8961018
> pxt(TD8890,90,.5,"constant force")
[1] 0.8900582
> pxt(TD8890,90,.5,"hyperbolic")
[1] 0.8840554
>
> pxtL <- function(u){pxt(TD8890,90,u,"linear")}; PXTL <- Vectorize(pxtL)
> pxtC <- function(u){pxt(TD8890,90,u,"constant force")}; PXTC <- Vectorize(pxtC)
> pxtH <- function(u){pxt(TD8890,90,u,"hyperbolic")}; PXTH <- Vectorize(pxtH)
> u=seq(0,1,by=.025)
> plot(u,PXTL(u),type="l")
> lines(u,PXTC(u),col="grey")
> lines(u,PXTH(u),pch=3,lty=2)
> points(c(0,1),PXTH(0:1),pch=19)
```



Linear
Constant force of mortality
Hyperbolic
Survival probability
1.00
0.95
0.90
0.85
0.80
0.0
0.2
0.4
0.6
0.8
1.0
Year

Interpolation of survival probabilities

The fist one is based on some linear interpolation between ${}_{\lfloor h \rfloor}p_x$ et ${}_{\lfloor h \rfloor+1}p_x$

$$ {}_h\tilde{p}_x = (1 - h + \lfloor h \rfloor)\ {}_{\lfloor h \rfloor}p_x + (h - \lfloor h \rfloor)\ {}_{\lfloor h \rfloor+1}p_x $$

For the second one, recall that ${}_hp_x = \exp\left(-\int_0^h \mu_{x+s}ds\right)$. Assume that $s \mapsto \mu_{x+s}$ is constant on $[0,1)$, then devient

$$ {}_hp_x = \exp\left(-\int_0^h \mu_{x+s}ds\right) = \exp[-\mu_x \cdot h] = (p_x)^h . $$

For the third one (still assuming $h \in [0,1)$), Baldacci suggested

$$ \frac{1}{{}_hp_x} = \frac{1 - h + \lfloor h \rfloor}{{}_{\lfloor h \rfloor}p_x} + \frac{h - \lfloor h \rfloor}{{}_{\lfloor h \rfloor+1}p_x} $$

or, equivalently ${}_hp_x = \dfrac{{}_{\lfloor h \rfloor+1}p_x}{1 - (1 - h + \lfloor h \rfloor)\ {}_{\lfloor h \rfloor+1\,h}q_x}$.

Deferred capital ${}_kE_x$, can be computed as

```
> Exn(TV8890,x=40,n=10,i=.04)
[1] 0.6632212
> pxt(TV8890,x=40,10)/(1+.04)^10
[1] 0.6632212
```

Annuities such as $\ddot{a}_{x:\overline{n|}}$'s or or $A_{x:\overline{n|}}$'s can be computed as

```
> Ex <- Vectorize(function(N){Exn(TV8890,x=40,n=N,i=.04)})
> sum(Ex(0:9))
[1] 8.380209
> axn(TV8890,x=40,n=10,i=.04)
[1] 8.380209
> Axn(TV8890,40,10,i=.04)
[1] 0.01446302
```

It is also possible to have *Increasing* or *Decreasing* (arithmetically) benfits,

$$IA_{x:\overline{n|}} = \sum_{k=0}^{n-1} \frac{k+1}{(1+i)^k} \cdot {}_{k-1}p_x \cdot {}_1q_{x+k-1},$$

or

$$DA_{x:\overline{n}|} = \sum_{k=0}^{n-1} \frac{n-k}{(1+i)^k} \cdot {}_{k-1}p_x \cdot {}_1q_{x+k-1},$$

The function is here

```
> DAxn(TV8890,40,10,i=.04)
[1] 0.07519631
> IAxn(TV8890,40,10,i=.04)
[1] 0.08389692
```

Note finally that it is possible to consider monthly benefits, not necessarily yearly ones,

```
> sum(Ex(seq(0,5-1/12,by=1/12))*1/12)
[1] 4.532825
```

In the `lifecontingencies` package, it can be done using the `k` value option

```
> axn(TV8890,40,5,i=.04,k=12)
[1] 4.532825
```

Consider an insurance where capital K if (x) dies between age x and $x+n$, and that the insured will pay an annual (constant) premium π. Then

$$K \cdot A_{x:\overline{m}|} = \pi \cdot \ddot{a}_{x:\overline{n}|}, \text{ i.e. } \pi = K \cdot \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

Assume that $x = 35$, $K = 100000$ and $= 40$, the benefit premium is

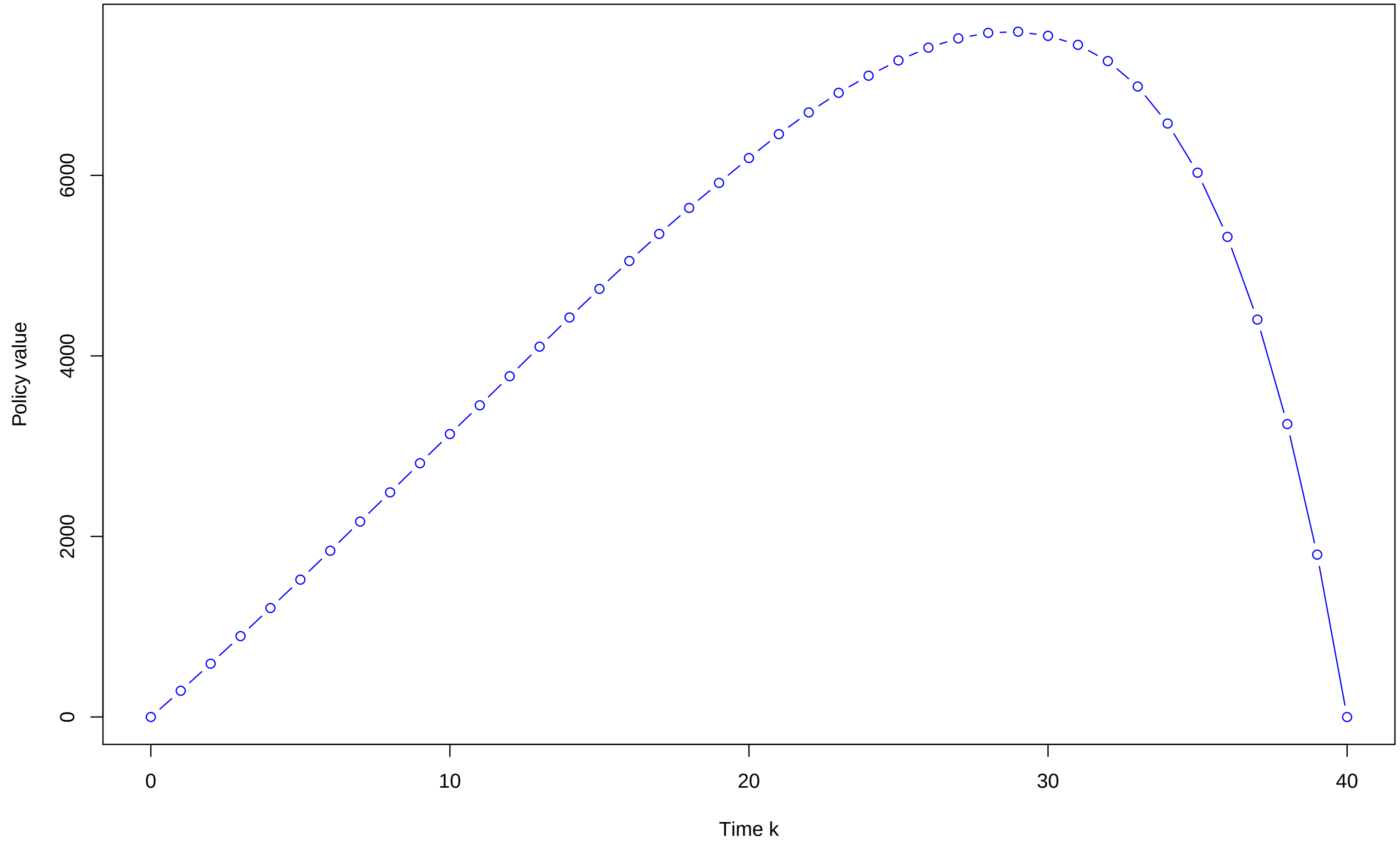
```
> (p <- 100000*Axn(TV8890,35,40,i=.04)/axn(TV8890,35,40,i=.04))
[1] 366.3827
```

For policy value, a prospective method yield

$${}_kV = K \cdot A_{x+k:\overline{n-k}|} - \pi \cdot \ddot{a}_{x+k:\overline{n-k}|}$$

i.e.

```
> V <- Vectorize(function(k){100000*Axn(TV8890,35+k,40-k,i=.04)-
+ p*axn(TV8890,35+k,40-k,i=.04)})
> V(0:5)
[1]    0.0000  290.5141  590.8095  896.2252 1206.9951 1521.3432
> plot(0:40,c(V(0:39),0),type="b")
```


Policy value
0
2000
4000
6000
0
10
20
30
40
Time k