Advanced trees in option pricing
(with some thoughts on teaching financial mathematics)

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Trees versus continuous diffusions

« The paper that showed that European option pricing could be put on a rational mathematical basis was Black and Scholes published in 1973. It was so revolutionary that the authors had to submit it to a number of journals before it was accepted. Although there are now numerous approaches to the result, they mostly require specialized methods, including Ito calculus and partial differential equations, and perhaps Girsanov theory and Feynman-Kac methods. But it is the binomial method due initially to Sharpe and substantially extended by Cox, Ross, and Rubinstein that made the theory of option pricing accessible to everyone with limited mathematical background. »

Trees versus continuous diffusions

« Even though it requires only routine algebraic manipulations, the method is still able to elucidate many of the ideas behind the full theory. Furthermore, all the surprising results mentioned in the opening can be located in this approach. For these reasons it is usually the first method presented in text books and finance courses; we shall follow this trend and step through it. The binomial method is, however, much more than a pedagogical breakthrough, since it allows for the development of numerical approximation methods for a wide range of options for which there are no known analytic solutions. »

Benchmark model in mathematical finance

Consider the standard diffusion for the price of a stock, under $\mathbb{P}$

$$
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \text{ or } S_1 = \begin{cases} 
S_u = S_0 \cdot u, \text{ with probability } p \\
S_d = S_0 \cdot d, \text{ with probability } 1 - p
\end{cases}
$$

with a nonrisky bond,

$$
\frac{dB_t}{B_t} = r dt \text{ or } B_1 = B_0 \cdot e^r
$$


without even mentioning Girsanov, Feyman-Kac...
Binomial trees in mathematical finance

Using replication techniques, and no-arbitrage assumption (law of one price), see


One period model, then

\[
C_0 = e^{-r} \left( \frac{e^r - d}{u - d} \cdot C_u + \frac{u - e^r}{u - d} \cdot C_d \right) = \frac{1}{1 + r} \mathbb{E}_{Q_1}(C_1).
\]

With a standard European call,

\[
C_0 = e^{-r} \left( \frac{(1 + r) - d}{u - d} \cdot [S_0 \cdot u - k]^+ + \frac{u - (1 + r)}{u - d} \cdot [S_0 \cdot d - k]^+ \right) = \frac{1}{1 + r} \mathbb{E}_{Q_1}([S_1 - k]^+).
\]
Binomial trees from 1 to $n$ periods

Recombining tree, $S_1$ takes values $S_0 \cdot u^i \cdot (1 - d)^{n-i}$, where $i = 0, \cdots, n$.

$$C_0 = e^{-r} \sum_{i=0}^{n} q^i (1 - q)^{n-i} \cdot [S_0 \cdot u^i \cdot (1 - d)^{n-i} - k]_+ = e^{-r} \mathbb{E}_{Q_n} (C_1).$$

where $q = \frac{e^{\frac{r}{n}} - d}{u - d}$, $u = \exp \left( \frac{\sigma}{\sqrt{n}} \right)$ et $d = \exp \left( -\frac{\sigma}{\sqrt{n}} \right)$. 

![Diagram of binomial tree](image)
Implied probabilities

$Q_n$ is the risk neutral probability. One can extract risk neutral probabilities (implied trees), from series of european call market prices, same maturity, different strikes $K_i$, using quadratic linear programming (with constraints)


even with different maturities


Implied volatility

Similarly, one can extract volatilites (implied volatility), from series of european call market prices,

Convergence property

Modifying the lattice to improve accuracy

Standard idea: use the property that, under $Q$,

$$S_{t+rac{1}{n}} = S_t \cdot e^{\frac{\sigma}{\sqrt{n}} Z}$$

where $Z$ takes values $\pm 1$ (with probabilities $q$ and $1-q$)

In standard trees, match first two moments, as $n \to \infty$


Use $u = e^{\frac{\sigma}{\sqrt{n}^2} + (r - \frac{\sigma^2}{2}) \frac{1}{n}}, d = e^{-\frac{\sigma}{\sqrt{n}} + (r - \frac{\sigma^2}{2}) \frac{1}{n}}$ and probability $1/2$,


Use $u = e^{\frac{\sigma}{\sqrt{n}} + \lambda \sigma^2 n}, d = e^{-\frac{\sigma}{\sqrt{n}} + \lambda \sigma^2 n}$ and probability $q$, for some tilt parameter $\lambda$ (extra degree of freedom). Here, perfect match of the first three moments exactly, for all $n$.

Modifying the lattice to improve accuracy

Use $u = e^{\lambda \frac{\sigma}{\sqrt{n}}}$, $d = e^{-\frac{1}{\lambda} \frac{\sigma}{\sqrt{n}}}$ and probability $q$, for some stretch parameter $\lambda$ (extra degree of freedom). This can be related to trinomial trees, $(u, 1, d)$.

To improve accuracy and computational efficiency,

- one can also distort the tree so that the strike lies half-way between two nodes
- one can smooth payoff functions at maturity

Numerical techniques in finance

- **Partial differential equation and numerical grids**

« binomial option pricing, suitably parametrized, is a special case of the explicit finite difference method »


- **Monte Carlo, and Quasi-Monte Carlo**


Need to define a lattice rule for Quasi-Monte Carlo methods

American options, backward induction for a put

Binomial trees for multiple assets

Consider a standard diffusion for the price of two stocks $S_t = (S_{1,t}, S_{2,t})$, under $\mathbb{P}$

$$
\frac{dS_t}{S_t} = \mu dt + \Sigma dW_t
$$

with a nonrisky bond,

$$
\frac{dB_t}{B_t} = r dt \text{ or } B_1 = B_0 \cdot e^r
$$

Binomial trees for multiple assets

\[ S_{1,t+\frac{1}{n}} = S_{1,t} \cdot e^{\frac{\Sigma_{11}}{\sqrt{n}} Z_1 + \left( r - \frac{\Sigma_{11}^2}{2} \right) \frac{1}{n}} \]

and

\[ S_{2,t+\frac{1}{n}} = S_{2,t} \cdot e^{\frac{\Sigma_{22}}{\sqrt{n}} Z_2 + \left( r - \frac{\Sigma_{22}^2}{2} \right) \frac{1}{n}} \]

where \( Z = (Z_1, Z_2) \) take values

\[
Z = \begin{bmatrix}
(+1,+1) & (+1,-1) \\
(-1,+1) & (-1,-1)
\end{bmatrix}
\]
Binomial trees for time varying - stochastic volatility


Change the number of steps such that the tree recombines,


\[
\frac{dS_t}{S_t} = \mu dt + h(\Sigma_t) dW_{1t} \quad \text{where} \quad \frac{d\Sigma_t}{\Sigma_t} =adt + bdW_{2t}
\]

**Binomial trees and stochastic interest rates**


**Binomial trees for barriers**

Binomial trees for path dependent assets

- **Lookback options**, e.g. \( \max\{S_t, t \in [0, 1]\} - S_1 \)

- **Asian options**, e.g. \( \int_0^1 S_t - k dt - k \)

Consider a more complex derivative, e.g. \( h(S) = \int_0^1 [S_t - k]_+ dt \)
Binomial trees for path dependent assets

Even with a recombining tree, the number of (distinct) paths grows exponentially with the number of steps. Recall that

$$\mathbb{E}_{Q_n}[h(S)] = \sum_{u \in \{0,1\}^n} Q_n(u) h(S_u).$$

Define

$$\{u \in \{0,1\}^n\} = \{u \in \{0,1\}^n; h(S_u) = 0\}
\cup \{u \in \{0,1\}^n; h(S_u) \neq 0\} = U_0 \cup U_+$$

$$\mathbb{E}_{Q_n}[h(S)] = \sum_{u \in U_+} \frac{Q_n(u)}{Q_n(U_+)} h(S_u)$$

Binomial trees and monte carlo

Generate $m$ trajectories $S_i$
under $\mathbb{Q}_n$, i.e. binomial vectors $U \in \{0, 1\}^n$

$$\frac{1}{m} \sum_{i=1}^{m} h(S_i^{\mathbb{Q}_n}) \to \mathbb{E}_{\mathbb{Q}_n}[h(S)]$$
as $m \to \infty$
Binomial trees and importance sampling

\[
\frac{1}{m} \sum_{i=1}^{m} \frac{\mathbb{P}(S_i)}{\mathbb{Q}_n(S_i)} h(S_i^\mathbb{P}) \rightarrow \mathbb{E}_{\mathbb{Q}_n}[h(S)] \quad \text{as} \quad m \rightarrow \infty
\]

Take home message

• (binomial) tree have interesting pedagogical virtues (change of measure, risk neutral valuation, implied probabilities, implied volatility, etc.)
• (binomial) tree can be used to price complex products (American, multiple assets, even path dependent)
• (binomial) tree have strong connexions with combinatorial arithmetics (generating vectors $\mathbf{u} \in \{a, b\}^n$ - with some constraints - can be done efficiently)