Advanced trees in option pricing

(with some thoughts on teaching financial mathematics)

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Trees versus continuous diffusions

« *The paper that showed that European option pricing could be put on a rational mathematical basis was Black and Scholes published in 1973. It was so revolutionary that the authors had to submit it to a number of journals before it was accepted. Although there are now numerous approaches to the result, they mostly require specialized methods, including Ito calculus and partial differential equations, and perhaps Girsanov theory and Feynman-Kac methods.* ***But it is the binomial method due initially to Sharpe and substantially extended by Cox, Ross, and Rubinstein that made the theory of option pricing accessible to everyone with limited mathematical background.*** »

- Price, J.F. (1996) Optional Mathematics Is Not Optional. *Notices of the A.M.S.*, **43**, 964-971

Trees versus continuous diffusions

« *Even though it requires only routine algebraic manipulations,* ***the method is still able to elucidate many of the ideas behind the full theory****. Furthermore, all the surprising results mentioned in the opening can be located in this approach. For these reasons it is usually the first method presented in text books and finance courses ; we shall follow this trend and step through it.* ***The binomial method is, however, much more than a pedagogical breakthrough, since it allows for the development of numerical approximation methods for a wide range of options for which there are no known analytic solutions****.* »

- Price, J.F. (1996) Optional Mathematics Is Not Optional. *Notices of the A.M.S.*, **43**, 964-971

Benchmark model in mathematical finance

Consider the standard diffusion for the price of a stock, under $\mathbb{P}$

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \text{ or } S_1 = \begin{cases} \mathrm{S}_u = S_0 \cdot u, \text{ with probability } p \\ \mathrm{S}_d = S_0 \cdot d, \text{ with probability } 1-p \end{cases}$$

with a nonrisky bond,

$$\frac{dB_t}{B_t} = rdt \text{ or } B_1 = B_0 \cdot e^r$$

- Black, F. & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* **81** 637-654.
- Sharpe, W.F. (1978). Investment, Prentice-Hall.
- Cox, J.C., Ross, S.A. & Rubinstein, M. (1979). Option Pricing : A Simplified Approach. *Journal of Financial Economics* **7**, 229-263.
- Rendleman, R.J. & Bartter, B.J. (1979). Two-State Option Pricing. *Journal of Finance* **34**, 1093-1110.

without even mentioning Girsanov, Feyman-Kac...

Binomial trees in mathematical finance

Using replication techniques, and no-arbitrage assumption (law of one price), see

- Arrow, K.J. & Debreu, G. (1954). Existence of an Equilibrium for a Competitive Economy. *Econometrica* **22**, 265-29.
- Merton, R.C. (1973). Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science* **4**, 141-183.
- Harrison, J.M. & Kreps, D.M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* **20**, 381-408.

One period model, then

$$C_0 = e^{-r}\left(\underbrace{\frac{e^r-d}{u-d}}_{q}\cdot C_u + \underbrace{\frac{u-e^r}{u-d}}_{1-q}\cdot C_d\right) = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}_1}(C_1).$$

With a standard European call,

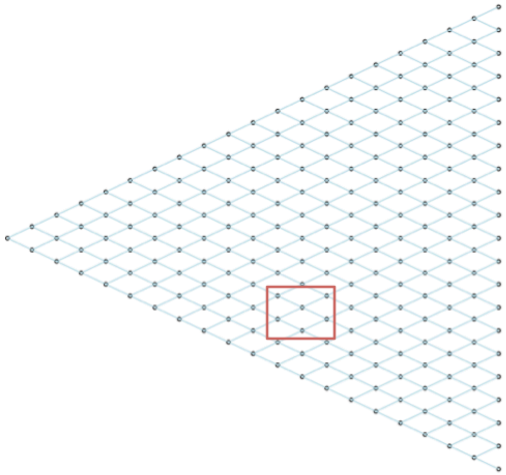
$$C_0 = e^{-r}\left(\underbrace{\frac{(1+r)-d}{u-d}}_{q}\cdot[S_0\cdot u-k]_+ + \underbrace{\frac{u-(1+r)}{u-d}}_{1-q}\cdot[S_0\cdot d-k]_+\right) = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}_1}([S_1-k]_+).$$

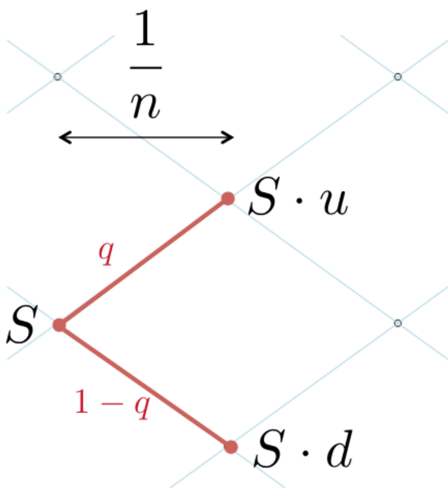
Binomial trees from 1 to n periods

Recombining tree, S_1 takes values $S_0 \cdot u^i \cdot (1-d)^{n-i}$, where $i = 0, \cdots, n$.

$$C_0 = e^{-r} \sum_{i=0}^{n} q^i (1-q)^{n-i} \cdot [S_0 \cdot u^i \cdot (1-d)^{n-i} - k]_+ = e^{-r} \mathbb{E}_{\mathbb{Q}_n}(C_1).$$

where $q = \dfrac{e^{\frac{r}{n}} - d}{u-d}$, $u = \exp\left(\dfrac{\sigma}{\sqrt{n}}\right)$ et $d = \exp\left(-\dfrac{\sigma}{\sqrt{n}}\right)$.





Implied probabilities

$\mathbb{Q}_n$ is the risk neutral probability. One can extract risk neutral probabilities (implied trees), from series of european call market prices, same maturity, different strikes K_i, using quadratic linear programming (with constraints)

- Rubinstein (1994) Implied binomial trees. *The Journal of Finance*, **49**, 771-818.
- Rubinstein (1995) As simple as one, two, three. *Risk*, **8**, 44-47.

even with different maturities

- Jackwerth, J.C. (1997). Generalized binomial trees.*The Journal of Derivatives*, **5**, 7-17

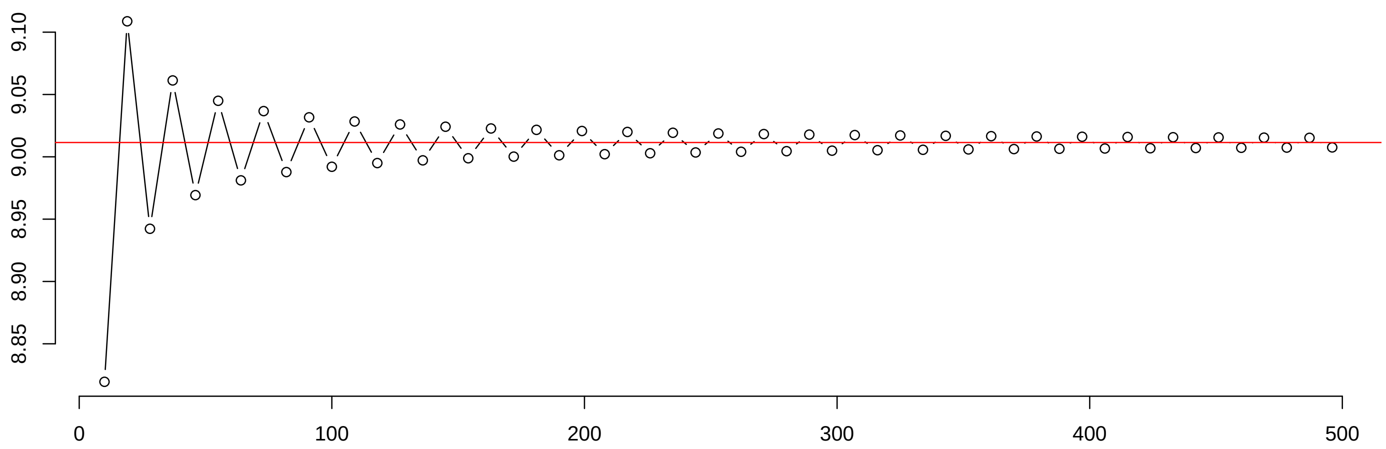
Implied volatility

Similarly, one can can extract volatilites (implied volatility) , from series of european call market prices,

- Barle, S. & Cakici, N. (1988) Growing a smiling tree. *The Journal of Financial Engineering*, **7**, 127-146.
- Derman, E. & Kani, I. (1994) Voltility smile and Its Implied Tree. *Quantitative Strategies Research Notes, Goldman Sachs.*
- Chriss, N. (1996). Transatlantic trees. *Risk*, **9**, 45-48.

Convergence property

- Cox, J.C., Ross, S.A. & Rubinstein, M. (1979). Option Pricing : A Simplified Approach. *Journal of Financial Economics* **7** : 229-263.
- Hsia, C.-C. (1983). On binomial option pricing. *Journal of Financial Research*, **6**, 41â46
- He, H. (1990) Convergence from discrete- to continuous-time contingent claims prices. *Review of Financial Studies*, **3**, 523-546.
- Li, A. (1992). Binomial approximation : computational simplicity and convergence. *Working Paper, Federal Reserve of Cleveland*, **9201**
- Nelson, D. B. and Ramaswamy, K. (1990). Simple binomial processes as diffusion approximations in financial models. Review of Financial Studies, 3 :393-430.
- Nelson, D. B. and Ramaswamy, K. (1990). Simple binomial processes as diffusion approximations in financial models. Review of Financial Studies, 3 :393-430.



Modifying the lattice to improve accuracy

Standard idea : use the property that, under $\mathbb{Q}$,

$$S_{t+\frac{1}{n}} = S_t \cdot e^{\frac{\sigma}{\sqrt{n}} Z} \text{ where } Z \text{ takes values } \pm 1 \text{ (with probabilities } q \text{ and } 1-q)$$

In standard trees, match first two moments, as $n \to \infty$

- Jarrow, R. & Rudd, A. (1983). Option pricing. Homewood.

Use $u = e^{\frac{\sigma}{\sqrt{n}} + \left(r - \frac{\sigma^2}{2}\right)\frac{1}{n}}$, $d = e^{-\frac{\sigma}{\sqrt{n}} + \left(r - \frac{\sigma^2}{2}\right)\frac{1}{n}}$ and probability $1/2$,

- Tian, Y.S. (1999). A flexible binomial option model. *Futures Markets*, **13**, 563-577.

Use $u = e^{\frac{\sigma}{\sqrt{n}} + \lambda\sigma^2 n}$, $d = e^{-\frac{\sigma}{\sqrt{n}} + \lambda\sigma^2 n}$ and probability q, for some tilt parameter λ (extra degree of freedom). Here, perfect match of the first three moments exactly, for all n.

- Brennan, M. & Schwartz, E. (1978) Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims : A Synthesis. *Journal of Financial and Quantitative Analysis* **13**, 461-474.
- Kamrad, B. & Ritchken, P. (1991) Multinomial Approximating Models for Options with k- State Variables. *Management Science*, **37**, 1640-1652.
- Chung S.-L. & Shih, P.-A. (2007). Generalized Cox-Ross-Rubinstein binomial models. *Management Science*, **53**, 508-520.

Modifying the lattice to improve accuracy

Use $u = e^{\lambda \frac{\sigma}{\sqrt{n}}}$, $d = e^{-\frac{1}{\lambda}\frac{\sigma}{\sqrt{n}}}$ and probability q, for some stretch parameter λ (extra degree of freedom). This can be related to trinomial trees, $(u, 1, d)$.

To improve accuracy and computational efficiency,

- one can also distort the tree so that the strike lies half-way between two nodes
- one can smooth payoff functions at maturity

- Brennan, M. & Schwartz, E. (1978) Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims : A Synthesis. *Journal of Financial and Quantitative Analysis* **13**, 461-474.
- Heston, S. & Zhou, G.. (2000) On the rate of convergence of discrete-time contingent claims. *Mathematical Finance*, **10** 53-75.
- Joshi, M.S. (2009). The convergence of binomial trees for pricing the American put. *The Journal of Risk*, **11**, 87-108.

Numerical techniques in finance

- **Partial differential equation and numerical grids**

« binomial option pricing, suitably parametrized, is a special case of the explicit finite difference method »

- Rubinstein, M. (2000). On the Relation Between Binomial and Trinomial Option Pricing Models. *Journal of Derivatives*, **8**, 47â50.

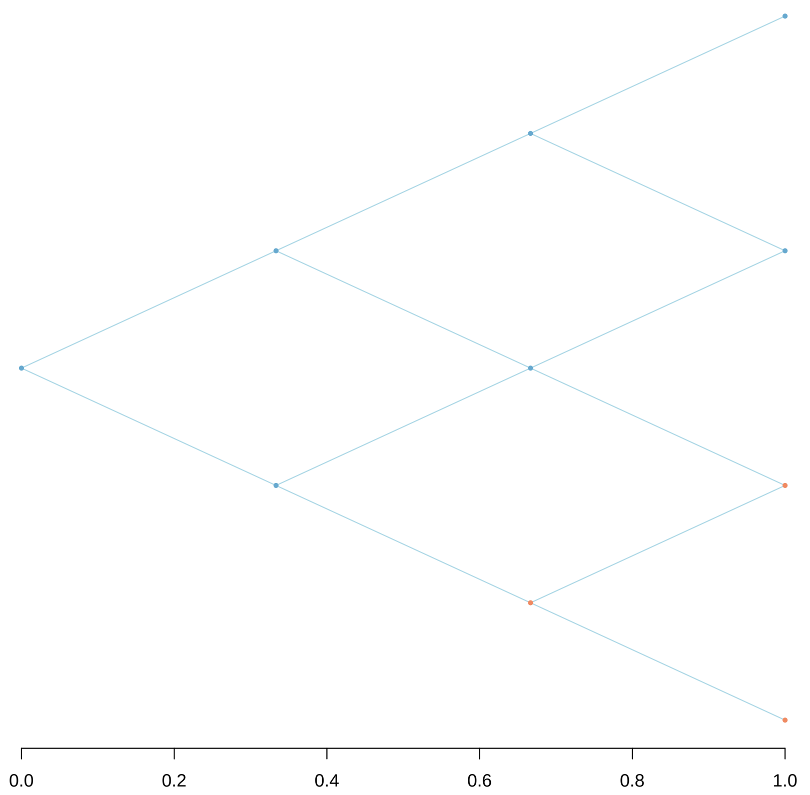
- **Monte Carlo, and Quasi-Monte Carlo**

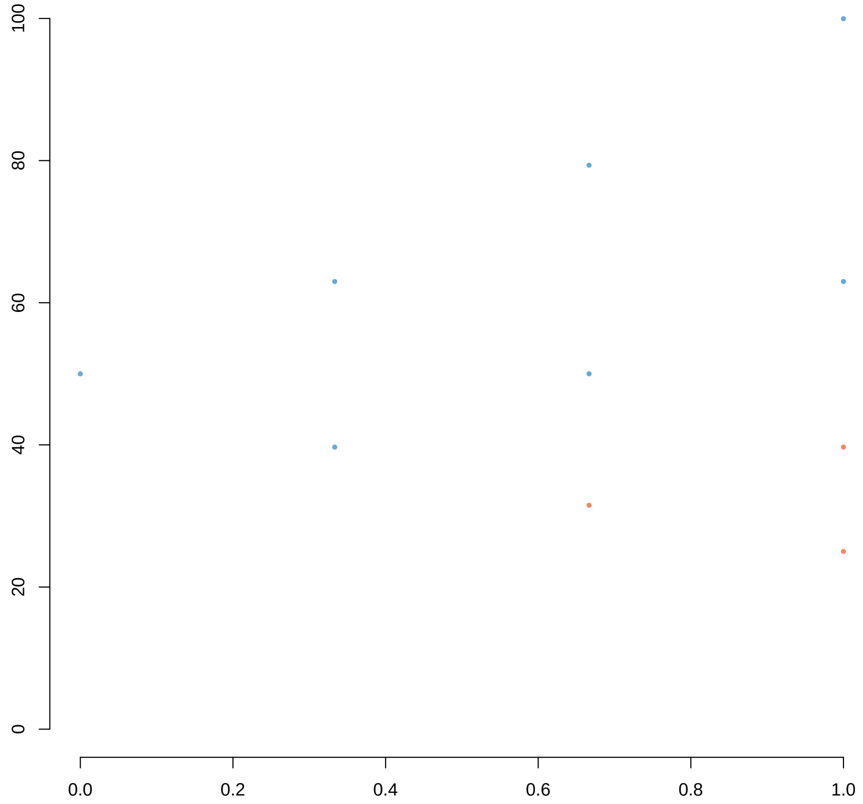
- Boyle, P. (1977), Options : a Monte Carlo approach, *Journal of Financial Economics*, **4**, 323-338.
- Paskov, S. H. and Traub, J. F. (1995), Faster evaluation of financial derivatives, *Journal of Portfolio Management*, **22**, 113-120.

Need to define a lattice rule for Quasi-Monte Carlo methods

- Boyle, P. (1977), Options : a Monte Carlo approach, J. Financial Economics, 4, 323-338.
- Boyle, P. (1988). A lattice framework for option pricing with two state variables. *Financial and Quantitative Analysis*, **23**, 1-12.
- Breen, R. (1991). The accelerated binomial option pricing model. Journal of Financial and Quantitative Analysis, 26(2) :153-164.
- Chen, R. and Yang, T. T. (1999). A universal lattice. Review of Derivative Research, 3 :115-133.
- Figlewski, S. and Gao, B. (1999). The adaptive mesh model : a new approach to efficient option pricing. Journal of Financial Economics, 53 :313-351.

American options, backward induction for a put





- Amin, K. & Khanna, A. (1994). Convergence of American option values from discrete to continuous time financial models. *Mathematical Finance*, **4** 289-304.
- Broadie, M. & Detemple, J. (1996) American Option Valuation : New Bounds, Approximations, and a Comparison of Existing Methods. *Review of Financial Studies*, 1996, Vol. 9, No. 4, pp. 1211-1250.

Binomial trees for multiple assets

Consider a standard diffusion for the price of two stocks $\boldsymbol{S}_t = (S_{1,t}, S_{2,t})$, under $\mathbb{P}$

$$\frac{d\boldsymbol{S}_t}{\boldsymbol{S}_t} = \boldsymbol{\mu} dt + \boldsymbol{\Sigma} d\boldsymbol{W}_t$$

with a nonrisky bond,

$$\frac{dB_t}{B_t} = r dt \text{ or } B_1 = B_0 \cdot e^r$$

- Boyle, P. P., Evnine, J., and Gibbs, S. (1989). Numerical evaluation of multivariate contingent claims. Review of Financial Studies, 2 :241-250.
- Madam, D. B., Milne, F., and Shefrin, H. (1989). The multinomial option pricing model and its Brownian and Poisson limits. Review of Financial Studies, 2(2) :251-265.
- Rubinstein, M. (1994). Return to Oz. *Risk Magazine*, **7**.
- Zhang P.G. (1995) Correlation Digital Options *Journal of Financial Engineering*, **3**, 5.

Binomial trees for multiple assets

$$S_{1,t+\frac{1}{n}} = S_{1,t} \cdot e^{\frac{\Sigma_{11}}{\sqrt{n}} Z_1 + \left(r - \frac{\Sigma_{11}^2}{2}\right)\frac{1}{n}} \text{ and } S_{2,t+\frac{1}{n}} = S_{2,t} \cdot e^{\frac{\Sigma_{22}}{\sqrt{n}} Z_2 + \left(r - \frac{\Sigma_{22}^2}{2}\right)\frac{1}{n}}$$

where $\boldsymbol{Z} = (Z_1, Z_2)$ take values

$$\boldsymbol{Z} = \left[\begin{array}{cc} (+1,+1) & (+1,-1) \\ (-1,+1) & (-1,-1) \end{array} \right]$$

Binomial trees for time varying - stochastic volatility

- Amin, K. I. (1991). On the computation of continuous time option prices using discrete approximations. Journal of Financial and Quantitative Analysis, 26, 477-495.

Change the number of steps such that the tree recombines,

- Ho, T. S., Stapleton, R. C., and Subrahmanyan, M. G. (1995). Multivariate binomial approximations for asset prices with nonstationary variance and covariance characteristics. *Review of Financial Studies*, **8**, 1125-1152.
- Hull, J. and White, A. (1987). The pricing of options on assets with stochastic volatilities. Journal of Finance, 42 :281-300.
- Amin, K. I. (1995). Option pricing trees. *Journal of Derivatives*, **2**, 34-46.
- Nelson, D.B. & Ramaswamy, K. (1990). Simple binomial processes as diffusion approximation in financial models. *Review of Financial Studies*, **3**, 393-430.

$$\frac{dS_t}{S_t} = \mu dt + h(\Sigma_t) dW_{1t} \text{ where } \frac{d\Sigma_t}{\Sigma_t} = a dt + b dW_{2t}$$

- Hilliard, J.E. & Schwartz, A.L. (1997). Pricing options on traded assets under stochastic interest rates and volatility : a Binomial approach. *Journal of Financial Engineering*, **6**, 281-305.

Binomial trees and stochastic interest rates

- Ho & Lee (1986). Term structure movement and pricing interest rate continent claims. Journal of Finance, 41, 1011-1029
- Ho & Lee (1990). Interest rate future options and interest rates options. The Financial Review, 25, 345-370
- Pederson, Shiu & Thorlacius (1990) Arbitrage free pricing pricing of interest rate contingent claims. Transaction of the SOA, 41, 231-279
- Li, A., Ritchken, P. & Sankarasubramamian. (1995) Lattice models for pricing american interest rate claims. *Journal of Finance*, **50**, 719-737.
- Hull, J. & White, A. (1990) Pricing interest-rate derivative securities *The Review of Financial Studies 3*, 573-592
- Hull, J. & White, A. (2000) Using Hull-White interest rate trees *Journal of Derivatives 3*, 26â36
- Lesne, J.P., Prigent, J.L. & Scaillet, O. (2000) Convergence of Discrete Time Options Pricing Models under Stochastic Rates. *Finance & Stochastics* **4**, 81-93.

Binomial trees for barriers

- Boyle, P.P. & Lau, S.H. (1994.) Bumping Up Against the Barrier with the Binomial Method. *The Journal of Derivatives*, **1**, 6-14.
- Ritchken, P.H. (1995). On Pricing Barrier Options. *The Journal of Derivatives*, **3**, 19-28.
- Levy, E. & Mantion, F. (1997) Discrete by nature. *Risk*, **10**, 74-77.

Binomial trees for path dependent assets

- **Lookback options**, e.g. $\max\{S_t, t \in [0,1]\} - S_1$

  - Boyle, P.P. & Lau, S.H. (1994.) Bumping Up Against the Barrier with the Binomial Method. *The Journal of Derivatives*, **1**, 6-14.
  - Ritchken, P.H. (1995). On Pricing Barrier Options. *The Journal of Derivatives*, **3**, 19-28.
  - Levy, E. & Mantion, F. (1997) Discrete by nature. *Risk*, **10**, 74-77.

- **Asian options**, e.g. $\left[\int_0^1 S_t - kdt - k\right]_+$

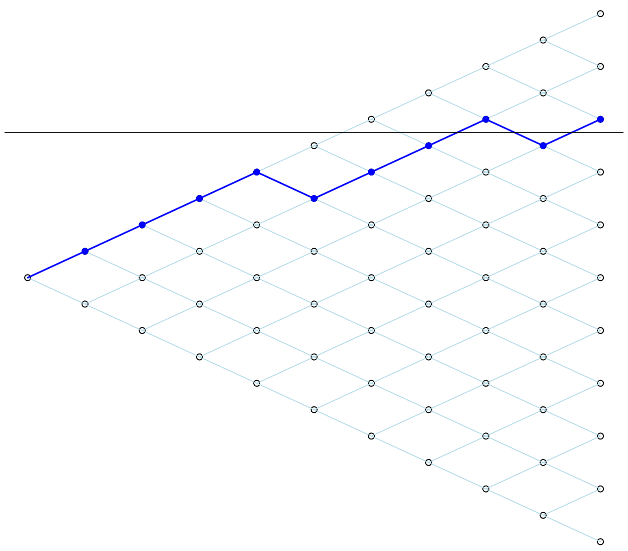
  - Hull, J., & White, A. (1993) Efficient Procedures for Valuing European and American Path-Dependent Options. *Journal of Derivatives*, **1**, 21-31.
  - Panjer, H.H. (1995) Exact pricing of geometric average options. *University of Waterloo, Technical Report*, **95-04**

Consider a more complex derivative, e.g. $h(\boldsymbol{S}) = \int_0^1 \left[S_t - k\right]_+ dt$

Binomial trees for path dependent assets

Even with a recombining tree, the number of (distinct) paths grows exponentially with the number of steps. Recall that

$$\mathbb{E}_{\mathbb{Q}_n}[h(\boldsymbol{S})] = \sum_{\boldsymbol{u}\in\{0,1\}^n} \mathbb{Q}_n(\boldsymbol{u})h(\boldsymbol{S}_{\boldsymbol{u}}).$$



Define

$$\left\{\boldsymbol{u} \in \{0,1\}^n\right\} = \left\{\boldsymbol{u} \in \{0,1\}^n; h(\boldsymbol{S}_{\boldsymbol{u}}) = 0\right\}$$
$$\cup\left\{\boldsymbol{u} \in \{0,1\}^n; h(\boldsymbol{S}_{\boldsymbol{u}}) \neq 0\right\} = \mathcal{U}_0 \cup \mathcal{U}_+$$

$$\mathbb{E}_{\mathbb{Q}_n}[h(\boldsymbol{S})] = \sum_{\boldsymbol{u}\in\mathcal{U}_+} \frac{\mathbb{Q}_n(\boldsymbol{u})}{\mathbb{Q}_n(\mathcal{U}_+)} h(\boldsymbol{S}_{\boldsymbol{u}})$$

- Ryser, H.J. (1963). Combinatorial Mathematics, Carus Monographs, MAA.
- Gale, D. (1957). A theorem on flows in networks, *Pacific Journal of Mathematics*, **7**, 1073-1082.

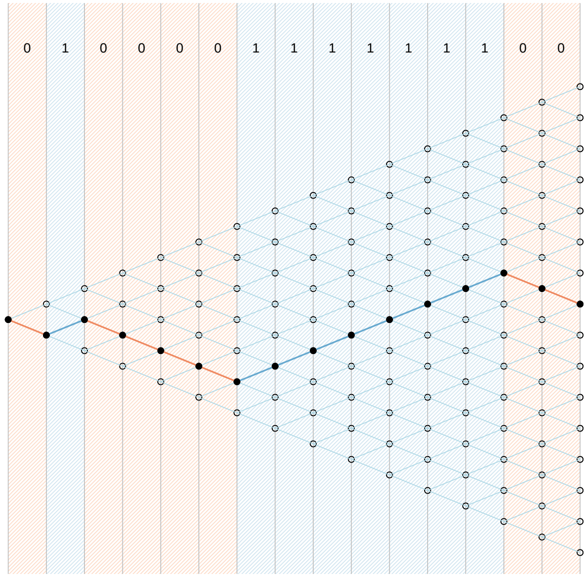
Binomial trees and monte carlo

Generate m trajectories $\boldsymbol{S}_i$

under $\mathbb{Q}_n$, i.e. binomial vectors $\boldsymbol{U} \in \{0,1\}^n$

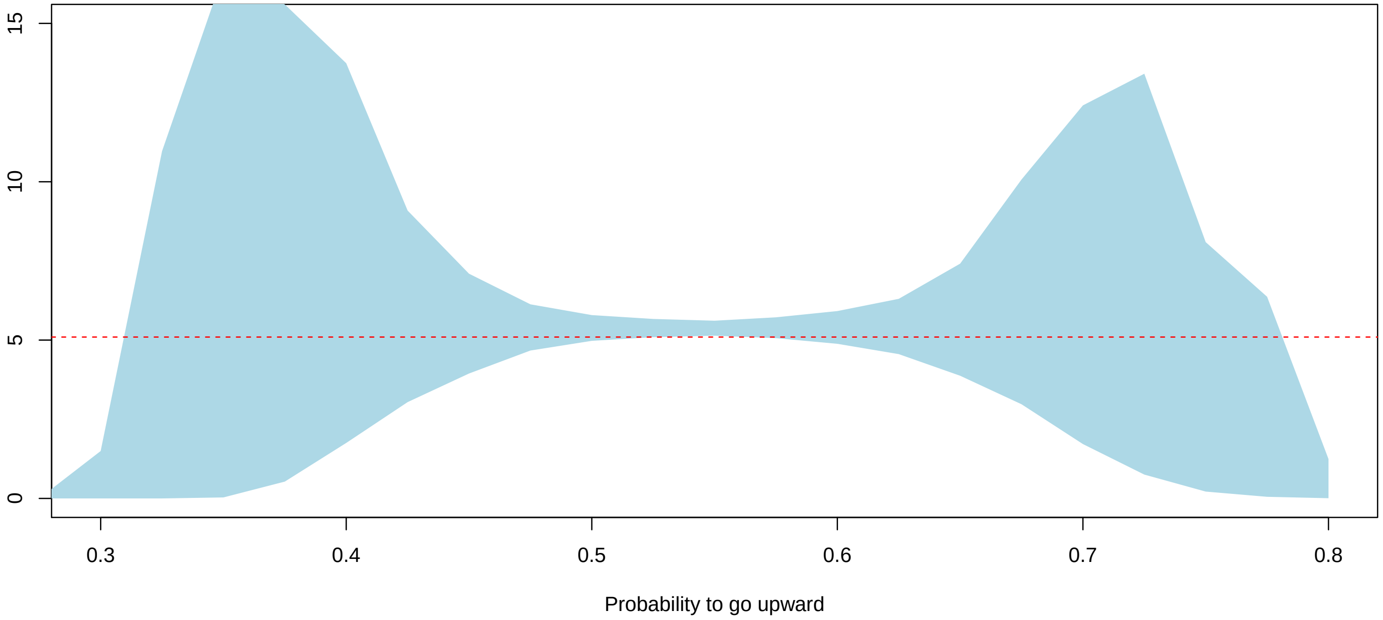
$$\frac{1}{m}\sum_{i=1}^{m} h(\boldsymbol{S}_i^{\mathbb{Q}_n}) \to \mathbb{E}_{\mathbb{Q}_n}[h(\boldsymbol{S})]$$

as $m \to \infty$



Binomial trees and importance sampling

$$\frac{1}{m}\sum_{i=1}^{m}\frac{\mathbb{P}(\boldsymbol{S}_i)}{\mathbb{Q}_n(\boldsymbol{S}_i)}h(\boldsymbol{S}_i^{\mathbb{P}}) \to \mathbb{E}_{\mathbb{Q}_n}[h(\boldsymbol{S})] \text{ as } m \to \infty$$



- Tan, K.S. (1998) Quasi-Monte Carlo methods : applications in finance and actuarial science. *University of Waterloo, PhD thesis.*
- Hörmann, W. & Leydold, J. (2005.) Quasi Importance Sampling *Technical Report, Institut für Statistik, WU Wien*

Take home message

- (binomial) tree have interesting pedagogical virtues (change of measure, risk neutral valuation, implied probabilities, implied volatility, etc.)
- (binomial) tree can be used to price complex products (American, multiple assets, even path dependent)
- (binomial) tree have strong connexions with combinatorial arithmetics

(generating vectors $\boldsymbol{u} \in \{a, b\}^n$ - with some constraints - can be done efficiently)